

SYMPLECTIC, POISSON, AND CONTACT GEOMETRY ON SCATTERING MANIFOLDS

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ABSTRACT. We introduce scattering-symplectic manifolds, manifolds with a type of minimally degenerate Poisson structure that is not too restrictive so as to have a large class of examples, yet restrictive enough for standard Poisson invariants to be computable.

This paper will demonstrate the potential of the scattering symplectic setting. In particular, we construct scattering-symplectic spheres and scattering symplectic gluings between strong convex symplectic fillings of a contact manifold. By giving an explicit computation of the Poisson cohomology of a scattering symplectic manifold, we introduce a new method of computing Poisson cohomology and apply it to b^k -symplectic manifolds.

1. Introduction

One example of a minimally degenerate Poisson structure is a *b-Poisson manifold*, defined by Victor Guillemin, Eva Miranda, and Ana Rita Pires [12] as a $2n$ -dimensional manifold M equipped with a Poisson bi-vector π that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\pi = x_1 \partial x_1 \wedge \partial y_1 + \sum_{i=2}^n \partial x_i \wedge \partial y_i.$$

We will study another type of minimally degenerate Poisson structure. Given a Poisson manifold (M, π) , if $(\wedge^n \pi)^{-1}(0)$ is an oriented hypersurface Z and J_Z denotes the ideal of functions vanishing at Z , if the associated Poisson bracket

$$\{\cdot, \cdot\} : \mathcal{C}^\infty(M) \times \mathcal{C}^\infty(M) \rightarrow \mathcal{C}^\infty(M)$$

is valued in J_Z and its restriction to $J_Z \times \mathcal{C}^\infty(M)$ is valued in J_Z^2 , then M is scattering-Poisson.

Using Richard Melrose's b-tangent bundle [21], Guillemin, Miranda, and Pires recover the correspondence of a non-degenerate Poisson bi-vector and a symplectic form; in this b-setting, a b-Poisson bi-vector corresponds to what

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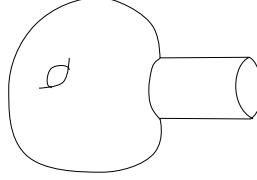
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is called a b-symplectic form. This correspondence allows us to understand the slightly degenerate b-Poisson structure using the language of symplectic geometry. Much work has been done to obtain the counterparts of results from symplectic geometry in the b-setting and to compute familiar Poisson invariants; for instance see [11, 12, 13], or [16].

Our goal is to similarly study minimally degenerate Poisson structures utilizing the tools of symplectic geometry. We desire a class of structures that is not overly restrictive so as to have a large class of examples, yet is restrictive enough to make the computation of Poisson invariants tractable.

The b-setting has its genesis in a compactification of a manifold with a cylindrical end in that the b-tangent bundle is a Lie algebroid that extends the standard tangent bundle to this compactification.

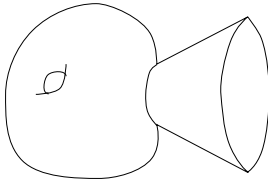
b - manifold



A torus with a cylindrical end

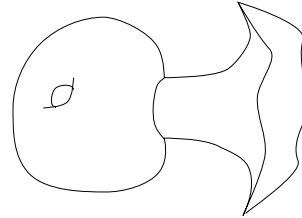
There are several other naturally occurring geometries on spaces whose compactifications also lend themselves to this endeavor of finding a class of minimally degenerate Poisson structures. We will discuss the scattering setting [24], a compactification of a manifold with a Euclidean end, and the 0-setting [18, 19], a compactification of a manifold with a hyperbolic-funnel end.

sc - manifold



A torus with a Euclidean end

0 - manifold



A torus with a hyperbolic-funnel end

In section 2 we describe Lie algebroids \mathcal{A} arising from this type of compactification, algebroids such as the b-tangent bundle, the 0-tangent bundle, and the sc-tangent bundle. An \mathcal{A} -symplectic form on a manifold M is a smooth form away from the boundary that extends to the Lie algebroid \mathcal{A} in the compactification. Sometimes there is a rich symplectic theory for a given algebroid \mathcal{A} , as has been the case for the b-tangent bundle. However, not all algebroids will

admit a symplectic form; we show that there are no 0-symplectic structures on manifolds of dimension greater than 2.

On the other hand, as we will show, a robust theory does exist for the sc-tangent bundle. For example, scattering-symplectic geometry includes the study of the standard Euclidean symplectic form at infinity. To be precise, let $(p_1, q_1, \dots, p_n, q_n)$ be the standard coordinates on \mathbb{R}^{2n} . Let

$$R = \sqrt{p_1^2 + q_1^2 + \dots + p_n^2 + q_n^2}$$

be the radial coordinate and away from the origin let $x = \frac{1}{R}$. Then in coordinates $t_i = p_i x$, $s_i = q_i x$ the standard symplectic form is expressible as

$$\omega = \sum_i dp_i \wedge dq_i = \sum_i \frac{dx}{x^3} \wedge (s_i dt_i - t_i ds_i) + \frac{1}{x^2} dt_i \wedge ds_i = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}$$

where $\alpha = s_i dt_i - t_i ds_i$ defines the standard contact structure on \mathbb{S}^{2n-1} . We compactify \mathbb{R}^{2n} with a sphere at infinity given by the zero set of x . This compactified space equipped with ω is an example of a scattering-symplectic manifold (with boundary).

In fact, any scattering-symplectic form ω on any manifold M will define a contact structure on the singular locus of ω . Accordingly, scattering-symplectic geometry not only gives us an approach to studying Poisson geometry, but contact geometry as well.

Additionally, scattering-symplectic geometry provides a way to recontextualize established results. In 1997, Peter Kronheimer and Tomasz Mrowka [14] adapted the construction of the Seiberg-Witten invariants to the situation of a 4-manifold X with contact boundary ∂X . To do this they construct a space X^+ that is the union of X with a cylinder $[1, \infty) \times \partial X$ by identifying ∂X with the closed end of the cylinder $\{1\} \times \partial X$. Given a contact form θ on ∂X , they define a symplectic form ω_0 on $[1, \infty) \times \partial X$ by the formula

$$\omega_0 = t dt \wedge \theta + \frac{1}{2} t^2 d\theta.$$

This cylinder naturally compactifies to a scattering-symplectic manifold with boundary. We identify $[1, \infty]$ with $[0, 1]$ by mapping $t \rightarrow \frac{1}{x}$ and ω_0 is expressible as the scattering-symplectic form

$$\frac{-dx}{x^3} \wedge \theta + \frac{d\theta}{2x^2}.$$

As a form on the sc-tangent bundle, this extends non-degenerately to $x = 0$.

The purpose of this paper is to introduce scattering-symplectic geometry and to demonstrate how this context can give new insights in symplectic, contact, and Poisson geometry. In the following subsections we give an account of select results.

1.1. Simple Poisson Structures on Spheres. Because every closed symplectic manifold has non-trivial degree 2 de Rham cohomology group, the only sphere that admits a symplectic structure is the 2 dimensional sphere \mathbb{S}^2 . An immediate question when expanding the notion of symplectic is to look for these structures on spheres.

Theorem 1.1. *Every even dimensional sphere \mathbb{S}^{2n} admits a scattering symplectic structure ω such that the equator \mathbb{S}^{2n-1} is the singular hypersurface and ω induces the standard contact structure on \mathbb{S}^{2n-1} .*

One very natural context that provides a generalized symplectic structure on spheres is folded symplectic geometry. Richard Melrose [23], by defining folded contact structures, introduced a particular idea of minimally degenerate differential form. Building on [23], Ana Cannas da Silva, Victor Guillemin, and Christopher Woodward [3] define a folded symplectic manifold (M^{2n}, Z, ω) to be a $2n$ -dimensional manifold M equipped with a closed two-form ω that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

By allowing this very mild degeneracy, all even dimensional spheres admit a folded symplectic structure. Folded symplectic geometry is similar in spirit to our main object of study (or to \mathcal{A} -symplectic structures). One significant difference is that \mathcal{A} -symplectic structures correspond to classical minimally-degenerate Poisson structures. The inverse of the morphism $\omega^b : TM \rightarrow T^*M$ (where it is invertible) for a folded symplectic form does not extend to define a Poisson structure on the entire manifold.

Unfortunately, there are no b-Poisson spheres in dimensions greater than two: Ioan Mărcuț, and Boris Osorno Torres [17] showed that a compact b -symplectic manifold (M, Z) of dimension $2n$ has a class c in $H^2(M)$ such that c^{n-1} is nonzero in $H^{2n-2}(M)$.

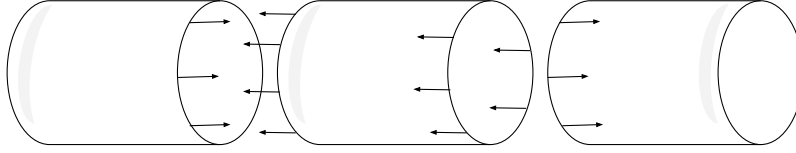
1.2. Symplectic Gluing. Recall that a symplectic manifold (M, ω) is a strong symplectic filling of a contact manifold (Z, ξ) if Z is the boundary of M and near Z there is a Liouville vector field V transverse to Z with $i_V \omega$ defining the contact structure ξ such that $\mathcal{L}_V \omega = \omega$. These fillings come in two flavors: *convex* means the Liouville vector field V points outward at the boundary Z and *concave* means V points inward at the boundary.

In order to glue two strong symplectic fillings along a common contact boundary to form a symplectic manifold, one side must be a concave filling and the other must be convex. In this paper, we demonstrate a natural way

to expand the symplectic category to allow gluings of convex to convex and concave to concave fillings.

Theorem 1.2. *Given two strong convex symplectic fillings of a contact manifold, their union over Z admits a scattering symplectic structure that coincides with the existing symplectic structures away from Z .*

Given two strong concave symplectic fillings of a contact manifold, their union over Z admits a folded symplectic structure that coincides with the existing symplectic structures away from Z .



On the left we glue convex to convex to produce a scattering symplectic union. On the right, concave to concave gives a folded symplectic union.

We can use this theorem to construct many examples of scattering-symplectic manifolds, see section 4.1. For instance, $\mathbb{T}^2 \times \mathbb{S}^2$ is scattering-symplectic with singular hypersurface three torus \mathbb{T}^3 . We also have that $\mathbb{S}^3 \times \mathbb{S}^1$ is scattering-symplectic with singular hypersurface $\mathbb{S}^2 \times \mathbb{S}^1$. While many scattering symplectic manifolds arise in this way, not all such structures can be obtained by gluing two strong convex fillings, see proposition 4.2.

1.3. Poisson Cohomology. The Poisson bi-vector determines a differential on multi-vector fields. This complex (\mathcal{V}^*, d_π) is the Lichnerowicz complex and its homology groups are Poisson cohomology.

Poisson cohomology is an important invariant in the study of Poisson structures. Unfortunately, the computation of Poisson cohomology is quite difficult in general and explicit results are known in only very select cases [5]. The simplest case is that of a symplectic manifold, where the Poisson bi-vector π is non-degenerate and the Poisson cohomology is isomorphic to the de Rham cohomology. The non-degeneracy of π allows us to define an isomorphism

$$TM \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp} \end{array} T^*M$$

that intertwines the respective differentials. This induces an isomorphism of complexes,

$$(\mathcal{V}^*, d_\pi) \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp} \end{array} (\Omega^*, d_{dR})$$

By realizing a Poisson bi-vector as the dual of an \mathcal{A} -symplectic form, we will have a similar isomorphism of complexes

$$({}^{\mathcal{A}}\mathcal{V}^*, d_\pi) \simeq ({}^{\mathcal{A}}\Omega^*, d)$$

where $({}^{\mathcal{A}}\mathcal{V}^*, d_\pi)$ is a subcomplex of the standard Lichnerowicz complex consisting of \mathcal{A} -multivector fields and $({}^{\mathcal{A}}\Omega^*, d)$ is the Lie algebroid cohomology of \mathcal{A} . However, in general the homology of these complexes does not compute the Poisson cohomology. In order to compute the Poisson cohomology of a scattering-symplectic manifold, we develop and present a new way of computing Poisson cohomology for this type of minimally degenerate Poisson structure inspired by this isomorphism. In certain cases of a Poisson manifold (M, π) , we can use π to define an isomorphism,

$$TM \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp} \end{array} \mathcal{R}^*$$

where \mathcal{R} is a Lie algebroid called the *rigged*¹ algebroid. This map intertwines the differential of the (usual) Lichnerowicz complex and the differential of the de Rham complex of \mathcal{R} , and thus induces an isomorphism of complexes

$$(\mathcal{V}^*, d_\pi) \begin{array}{c} \xrightarrow{\omega^\flat} \\ \xleftarrow{\pi^\sharp} \end{array} (\mathcal{R}\Omega^*, d_{\mathcal{R}})$$

that allows us to use Lie algebroid cohomology to compute Poisson cohomology. We use this method to compute two cases, the first being of a scattering-symplectic manifold. For simplicity, we state the result here using a fixed tubular neighborhood of Z . An invariant version can be found in Theorem 5.9.

Given a contact structure ξ on Z , let

$$\Omega_\xi^k(Z) := \left\{ \sigma \in \Omega^k(Z) \mid \forall p \in Z, \text{supp}(\sigma_p) \subseteq \wedge^k \xi_p \right\}$$

and let α be a contact form on Z such that $\ker \alpha = \xi$.

Theorem 1.3. *If (M, π) is a scattering-Poisson manifold, with induced contact structure ξ on Z , then the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z) \oplus \Omega_\xi^{p-1}(Z) \oplus \ker(d\alpha \wedge : \Omega_\xi^{p-2}(Z) \rightarrow \Omega_\xi^p(Z)).$$

Finally, we compute the Poisson cohomology of a b^k -symplectic manifold, a type of Poisson manifold introduced by Geoffrey Scott [25]. A b^k -Poisson manifold is an $2n$ -dimensional manifold M equipped with a Poisson bi-vector

¹The name is inspired by rigged Hilbert spaces from functional analysis.

π that is non-degenerate except on a hypersurface Z where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\pi = x_1^k \partial x_1 \wedge \partial y_1 + \sum_{i=2}^n \partial x_i \wedge \partial y_i.$$

Each b^k -symplectic structure induces a cosymplectic structure (θ, η) on Z . The flow of the Reeb vector field associated to the cosymplectic structure defines a foliation on Z , which we will denote \mathcal{F}_R . Consider the horizontal forms on this foliation:

$$\Omega_h^p(Z) = \{\sigma \in \Omega^p(Z) \mid i_R \sigma = 0\}.$$

We define an exterior derivative

$$d_h = d - \theta \wedge \mathcal{L}_R.$$

This forms a complex and we call its homology groups $H_h^*(\mathcal{F}_R)$ the *horizontal foliation cohomology of \mathcal{F}_R* .

Theorem 1.4. *If (M, π) is a b^k -Poisson manifold, then the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1}$$

for $k \geq 2$.

For $k = 1$, our method recovers the result of Ioan Mărcuț and Boris Osorno Torres [16] that

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

Sections 2-6 of this paper will provide the details and proofs of these results along with discussion of other topics. In section 2 we introduce Lie algebroids of interest, discuss symplectic and Poisson structures on these, and compute the Lie algebroid de Rham cohomology of certain examples. In section 3 we use symplectic techniques to discuss in detail the structure of a scattering-symplectic manifold in the neighborhood of Z , and provide the construction of scattering-symplectic spheres. We provide a discussion of contact hypersurfaces in section 4 and construct symplectic gluings. In section 5 we discuss the Poisson geometry of a scattering-symplectic manifold and compute Poisson cohomology. Certain technical details appear in section 6.

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Index of Notation and Common Terms

$\Gamma(E)$	Smooth sections of a vector bundle $E \rightarrow M$.
$(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$	A Lie algebroid over a manifold M , that is, a triple consisting of a vector bundle $\mathcal{A} \rightarrow M$, a Lie bracket $[\cdot, \cdot]_{\mathcal{A}}$ on the $\mathcal{C}^\infty(M)$ -module of sections $\Gamma(\mathcal{A})$, and a bundle map $\rho_{\mathcal{A}} : \mathcal{A} \rightarrow TM$ such that $[X, fY] = \mathcal{L}_{\rho_{\mathcal{A}}(X)}f \cdot Y + f[X, Y]$ where $X, Y \in \Gamma(\mathcal{A})$, and $f \in \mathcal{C}^\infty(M)$.
${}^{\mathcal{A}}\Omega^k(M)$	The set $\Gamma(\wedge^k \mathcal{A}^*)$ of smooth sections of the k -th exterior power of the dual bundle to \mathcal{A} , called the \mathcal{A} -de Rham forms on M .
${}^{\mathcal{A}}\mathcal{V}^k(M)$	The set $\Gamma(\wedge^k \mathcal{A})$ of smooth sections of the k -th exterior power of \mathcal{A} , together with d_π this is called the \mathcal{A} -Lichnerowicz complex.
(M, Z)	A manifold M and a compact hypersurface $Z \subset M$.
Z defining function	A defining function for a hypersurface $Z \subset M$, usually denoted x . That is, $x \in \mathcal{C}^\infty(M)$ such that $Z = \{p \in M : x(p) = 0\}$ and $dx(p) \neq 0$ for all $p \in Z$.
bTM	b-tangent bundle over a pair (M, Z) , the vector bundle whose sections are the vector fields on M that are tangent to Z at Z .
0TM	0-tangent bundle over a pair (M, Z) , the vector bundle whose sections are the vector fields on M that are zero at Z .
${}^{sc}TM$	Scattering-tangent bundle over a pair (M, Z) , the vector bundle whose sections are the sections of ${}^bTM \rightarrow M$ that are zero at Z .
(Z, ξ)	A hypersurface equipped with a contact structure ξ .

2. \mathcal{A} -Manifolds

To define b -symplectic geometry, Victor Guillemin, Eva Miranda, and Ana Rita Pires [12] make use of the b -tangent bundle introduced by R. Melrose [21]. We will make the analogous modification to the scattering tangent bundle presented by Melrose in [22]. We begin in what has previously been labeled the b -category.

Definition 2.1. [12] A b -manifold is a pair (M, Z) consisting of an oriented manifold M and an oriented hypersurface $Z \subset M$. A b -map is a map

$$f : (M_1, Z_1) \rightarrow (M_2, Z_2)$$

transverse to Z_2 such that $f^{-1}(Z_2) = Z_1$. The b -category is the category whose objects are b -manifolds and morphisms are b -maps.

As we will see, depending on context, we will have occasion to call such a (M, Z) a scattering-, zero-, or most generally an \mathcal{A} -manifold.

2.1. Rescaling Lie algebroids. Richard Melrose explained how to rescale a vector bundle with a filtration over a hypersurface; see Proposition 8.1 in [21]. We adapt his construction to Lie algebroids. We will describe how to rescale a Lie algebroid with respect to a suitable subbundle over a hypersurface.

Let (M, Z) be a manifold M with hypersurface $Z \subset M$ and let $(\mathcal{A}, \rho, [\cdot, \cdot])$ be a Lie algebroid over M . Given a subbundle $F \subseteq \mathcal{A}|_Z \rightarrow Z$, suppose that the sections of F are closed under the Lie bracket and the image of F under the anchor map is a subbundle of TZ :

$$[\Gamma(F), \Gamma(F)] \subseteq \Gamma(F) \text{ and } \rho(F) \subseteq TZ \rightarrow Z \text{ is a subbundle.} \quad (2.1)$$

Consider the space of sections:

$$\mathcal{D} = \{u \in \mathcal{C}^\infty(M; \mathcal{A}) : u|_Z \in \mathcal{C}^\infty(Z; F)\}. \quad (2.2)$$

\mathcal{D} consists of the set of smooth sections of \mathcal{A} that take values in F at Z .

If (2.1) holds, then there exists an algebroid $({}^F\mathcal{A}, \rho_F, [\cdot, \cdot]_F)$ over M whose space of sections ‘is’ \mathcal{D} as defined in (2.2).

Theorem 2.2. *There exists a vector bundle ${}^F\mathcal{A} \rightarrow M$ with an injective vector bundle map*

$$i : {}^F\mathcal{A} \rightarrow \mathcal{A}$$

that is an isomorphism over $M \setminus Z$ such that $i_\mathcal{C}^\infty(M; \mathcal{A}) = \mathcal{D}$. Let*

$$\rho_F : {}^F\mathcal{A} \rightarrow TM$$

be defined as $\rho_F = \rho \circ i$ and

$$[\cdot, \cdot]_F : \Gamma({}^F\mathcal{A}) \times \Gamma({}^F\mathcal{A}) \rightarrow \Gamma({}^F\mathcal{A})$$

be defined as $[\cdot, \cdot]_F = i^{-1}([i(\cdot), i(\cdot)])$. Then $({}^F\mathcal{A}, \rho_F, [\cdot, \cdot]_F)$ is a Lie algebroid over M .

Proof. We begin by noting that \mathcal{D} is preserved under multiplication by any smooth function on M and thus is a $\mathcal{C}^\infty(M)$ module. For any $p \in M$, we will consider the ideal $\mathcal{I}_p := \{f \in \mathcal{C}^\infty(M) : f(p) = 0\}$. Let

$${}^F\mathcal{A}_p = \mathcal{D}/(\mathcal{I}_p \cdot \mathcal{D}) \quad \text{and} \quad {}^F\mathcal{A} = \bigsqcup_{p \in M} {}^F\mathcal{A}_p.$$

Then because we can describe \mathcal{A}_p as $\mathcal{C}^\infty(M; \mathcal{A})/(\mathcal{I}_p \cdot \mathcal{C}^\infty(M; \mathcal{A}))$, there exists a natural map $i : {}^F\mathcal{A}_p \rightarrow \mathcal{A}_p$ taking a section in \mathcal{D} and evaluating it at the point p . For all $p \in M \setminus Z$, this map is an isomorphism.

Suppose $p \in Z$ and let $\{v_1, \dots, v_k\}$ be a local basis of smooth sections of F . We can smoothly extend this to a basis $\{v_1, \dots, v_k, v_{k+1}, \dots, v_M\}$ of \mathcal{A} . Given a Z defining function x , any element $X \in \mathcal{D}$ is locally of the form

$$X = \sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j$$

for smooth functions g_i . Thus $\{v_1, \dots, v_k, xv_{k+1}, \dots, xv_M\}$ is a local basis of ${}^F\mathcal{A}$ and the coefficients g_1, \dots, g_M give a local trivialization. Given any other local basis $\{v'_1, \dots, v'_k\}$ for F , we can extend smoothly to a local basis $\{v'_1, \dots, v'_k, v'_{k+1}, \dots, v'_M\}$ of \mathcal{A} . Each of $\{v'_1, \dots, v'_k, xv'_{k+1}, \dots, xv'_M\}$ can be expressed as a smooth linear combination of $\{v_1, \dots, v_k, xv_{k+1}, \dots, xv_M\}$. So this induces smooth transformations among the coefficients g_i and ${}^F\mathcal{A}$ inherits a natural smooth bundle structure from \mathcal{A} with bundle map $i : {}^F\mathcal{A} \rightarrow \mathcal{A}$. It is clear by construction that i is injective.

Given $(\mathcal{A}, [\cdot, \cdot], \rho)$, we can consider the rescaled vector bundle ${}^F\mathcal{A}$ as a Lie algebroid by taking anchor map ρ_F to be the composition of the bundle map $i : {}^F\mathcal{A} \rightarrow \mathcal{A}$ with bundle map $\rho : \mathcal{A} \rightarrow TM$ and by taking as a Lie bracket $[\cdot, \cdot]_F = i^{-1}([i(\cdot), i(\cdot)])$, the bracket induced from the bracket $[\cdot, \cdot]$ on \mathcal{A} . We must verify that $[\cdot, \cdot]_F$ is a well-defined map $\Gamma({}^F\mathcal{A}) \times \Gamma({}^F\mathcal{A}) \rightarrow \Gamma({}^F\mathcal{A})$, where $\Gamma({}^F\mathcal{A})$ is the $\mathcal{C}^\infty(M)$ -module of smooth sections of ${}^F\mathcal{A} \rightarrow M$.

As described above, given a Z defining function x , any elements $X, Y \in \mathcal{D}$ are locally of the form

$$X = \sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j \quad \text{and} \quad Y = \sum_{i=1}^k f_i v_i + \sum_{j=k+1}^M x f_j v_j.$$

Then $[X, Y]_F =$

$$\sum_{i,j=1}^k [g_i v_i, f_j v_j] + \sum_{i=1}^k \sum_{j=k+1}^M [g_i v_i, x f_j v_j] + \sum_{i=1}^k \sum_{j=k+1}^M [x g_i v_i, f_j v_j] + \sum_{i,j=k+1}^M [x g_i v_i, x f_j v_j].$$

It suffices to show that each term of the sum restricts to a section of $\Gamma(F)$ or vanishes at Z . For $1 \leq i, j \leq k$, we have that $[g_i v_i, f_j v_j]|_Z \in \Gamma(F)$ by assumption. Thus $[g_i v_i, f_j v_j] \in \Gamma({}^F\mathcal{A})$. Consider $[g_i v_i, x f_j v_j] = g_i \rho(v_i)(x) \cdot f_j v_j + x[g_i v_i, f_j v_j]$. Since $\rho(F)|_Z \subseteq TZ$, $\rho(v_i)(x) = 0$ which implies $[g_i v_i, x f_j v_j] \in \Gamma({}^F\mathcal{A})$. By antisymmetry of the Lie bracket, $[x g_i v_i, f_j v_j] \in \Gamma({}^F\mathcal{A})$. Finally consider $[x g_i v_i, x f_j v_j] = x g_i \rho(v_i)(x) \cdot f_j v_j + x[x g_i v_i, f_j v_j] \in \Gamma({}^F\mathcal{A})$.

To conclude, we must check that $[\cdot, \cdot]_F$ satisfies the Leibniz rule:

$$[X, fY]_F = \rho_F(X)f \cdot Y + f[X, Y]_F$$

where $X, Y \in \Gamma({}^F\mathcal{A})$, $f \in C^\infty(M)$ and $\rho_F(X)f$ is the Lie derivative of f with respect to the vector field $\rho_F(X)$. Consider $[X, fY]_F = i^{-1}([i(X), i(fY)]) =$

$$i^{-1}(\rho(i(X))f \cdot i(Y) + f[i(X), i(Y)]) = \rho_F(X)f \cdot Y + f[X, Y]_F$$

because i is an injective bundle map on $M \setminus Z$ and this identity extends by continuity. \square

Note that the restriction of ${}^F\mathcal{A}$ to Z is not F , but rather is a vector bundle of the same rank as \mathcal{A} . As explained by Melrose in Lemma 8.5 of [21], ${}^F\mathcal{A}|_Z$ is isomorphic to the graded bundle

$$F \oplus (N^*Z \otimes \mathcal{A}|_Z/F). \quad (2.3)$$

The conormal bundle here makes this bundle invariant of choice of Z defining function. Further, ${}^F\mathcal{A}$ is, non-canonically, isomorphic to \mathcal{A} : Given a Z defining function x , we can map a local expression of an element of $\Gamma({}^F\mathcal{A})$ to an element in $\Gamma(\mathcal{A})$ by

$$\sum_{i=1}^k g_i v_i + \sum_{j=k+1}^M x g_j v_j \rightarrow \sum_{i=1}^k g_i v_i + \frac{1}{x} \sum_{j=k+1}^M x g_j v_j.$$

Next, we will explore some specific applications of this construction.

Example 2.3. (Hyperbolic Geometry) We can consider the tangent bundle of a manifold $TM \rightarrow M$ as a Lie algebroid with Lie bracket the standard bracket on vector fields and with anchor map the identity map. We can apply Theorem 2.2 to a pair (M, Z) and rescale TM using the subbundle $0 \rightarrow Z$. The rescaled bundle ${}^0TM \rightarrow M$ is called the **zero tangent bundle** and was introduced by Rafe Mazzeo and Richard Melrose in the context of manifolds with boundary [18, 19]. In this case \mathcal{D} is the set of vector fields that vanish at Z . If x, y_1, \dots, y_n are local coordinates near a point in Z , and x is a defining function for Z , then the vector fields

$$x \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_n}$$

form a local basis for 0TM . Note that these do not vanish at Z as sections of 0TM . We call the dual bundle ${}^0T^*M \rightarrow M$ the **zero cotangent bundle**. This bundle is locally generated by

$$\frac{dx}{x}, \frac{dy_1}{x}, \dots, \frac{dy_n}{x}.$$

The anchor map of the 0TM algebroid is inclusion into the tangent bundle and the bracket is induced by the standard Lie bracket on TM .

To see the relation with hyperbolic geometry, let us consider the half-plane model of hyperbolic space and its associated metric:

$$\mathbb{H}^n = \{(x, y_1, \dots, y_n) \in \mathbb{R}^n \mid x > 0\} \quad g = \frac{dx^2}{x^2} + \frac{\sum_i dy_i^2}{x^2}.$$

We can create a new space, denoted $\bar{\mathbb{H}}^n$, by adding

$$Z = \{(x, y_1, \dots, y_n) \in \mathbb{R}^n \mid x = 0\}.$$

The vector fields of bounded pointwise length with respect to g are precisely the sections of ${}^0T\bar{\mathbb{H}}^n$, the rescaling of $T\bar{\mathbb{H}}^n$ by the 0 bundle over $Z = \{x = 0\}$. So the hyperbolic metric is naturally interpreted as a metric on ${}^0T\bar{\mathbb{H}}^n$. For a more complete discussion of the 0-tangent bundle and its role in hyperbolic geometry, see Section 8.3 of [22].

Example 2.4. (Cylindrical Geometry) We recover the **b -tangent bundle** as formulated in [12], by applying Theorem 2.2 to the tangent bundle TM over a pair (M, Z) , and taking as subbundle $TZ \rightarrow Z$.

The b -tangent bundle is the vector bundle whose space of sections is $\mathcal{D} = \{u \in \mathcal{C}^\infty(M; TM) : i \circ u|_Z \in \mathcal{C}^\infty(Z; TZ)\}$, the vector fields that are tangent to Z . If x, y_1, \dots, y_n are local coordinates near a point in Z , and x is a defining function for Z , then the vector fields and co-vectors

$$x \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n} \quad \frac{dx}{x}, dy_1, \dots, dy_n$$

respectively form local bases for bTM and ${}^bT^*M$. The anchor map of the bTM algebroid is inclusion into the tangent bundle and the bracket is induced by the standard Lie bracket on TM .

To see the relation with cylindrical geometry, given any compact Riemannian manifold (M, g_M) , let us consider the cylinder and metric:

$$C = \mathbb{R}_t \times M \quad g_C = dt^2 + g_M.$$

Consider the new coordinate $x = e^{-t}$. Then

$$g_C = \frac{dx^2}{x^2} + g_M.$$

We can create a new space, denoted \bar{C} , by adding the point $t = \{\infty\}$ (i.e. $x = \{0\}$) to the real line. Then the vector fields of bounded pointwise length with respect to g_C are precisely the sections of ${}^bT\bar{C}$, the rescaling of $T\bar{C}$ by the

bundle TM at $\{x = 0\} \times M$. So the cylindrical metric is naturally interpreted as a metric on ${}^bT\overline{C}$. For a more complete discussion of the b -tangent bundle and its role in cylindrical geometry, see Chapter 7 of [22].

The next construction, the scattering tangent bundle, provides our primary object of study.

Example 2.5. (Euclidean Geometry) We can apply Theorem 2.2 to the b -tangent bundle ${}^bTM \rightarrow M$ over a pair (M, Z) , and rescale using the subbundle $0 \rightarrow Z$. The resulting bundle ${}^{sc}TM \rightarrow M$ is called the **scattering tangent bundle**. If x is a defining function for Z , and y_1, \dots, y_n are local coordinates in Z , then the vector fields

$$x^2 \frac{\partial}{\partial x}, x \frac{\partial}{\partial y_1}, \dots, x \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{sc}TM$. The dual bundle ${}^{sc}T^*M \rightarrow M$ is called the **scattering cotangent bundle** and is locally generated by

$$\frac{dx}{x^2}, \frac{dy_1}{x}, \dots, \frac{dy_n}{x}.$$

The anchor map is inclusion into the b -tangent bundle and then into the tangent bundle TM . In the same way, the bracket is induced by the standard Lie bracket on TM .

To see the relation with Euclidean geometry, we consider Euclidean space with its standard metric. By performing the spherical compactification described in the Introduction, near the boundary this metric is

$$g = \frac{dx}{x^4} + \frac{g_S}{x^2}$$

where g_S is the standard metric on the sphere. The vector fields of bounded pointwise length with respect to g are the sections of the scattering-tangent bundle ${}^{sc}T\overline{\mathbb{R}}^n$ and g is naturally interpreted as a metric on ${}^{sc}T\overline{\mathbb{R}}^n$. For more details, see Section 1.8 of [22].

The next example is a generalization of the scattering tangent bundle that is comparable to Geoffrey Scott's b^k generalization of the b -tangent bundle [25].

Example 2.6. Consider the scattering tangent bundle ${}^{sc}TM$ associated to (M, Z) . The scattering-2 tangent bundle ${}^{sc^2}TM$ is the vector bundle whose space of sections is $\mathcal{D} = \{u \in \mathcal{C}^\infty(M; {}^{sc}TM) : u|_Z = 0\}$. In other words, ${}^{sc^2}TM$ is rescaling ${}^{sc}TM$ by the subbundle $0 \rightarrow Z$.

In this fashion, given the scattering- $(k-1)$ tangent bundle ${}^{sc^{(k-1)}}TM$ associated to (M, Z) , the **scattering- k tangent bundle** ${}^{sc^k}TM$ is the vector bundle whose space of sections is $\mathcal{D} = \left\{u \in \mathcal{C}^\infty(M; {}^{sc^{(k-1)}}TM) : u|_Z = 0\right\}$. If

x is a defining function for Z , and y_1, \dots, y_n are local coordinates in Z , then the vector fields

$$x^{k+1} \frac{\partial}{\partial x}, x^k \frac{\partial}{\partial y_1}, \dots, x^k \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{sc^k}TM$. The dual bundle ${}^{sc^k}T^*M \rightarrow M$ is called the ***scattering- k cotangent bundle*** and is locally generated by

$$\frac{dx}{x^{k+1}}, \frac{dy_1}{x^k}, \dots, \frac{dy_n}{x^k}.$$

For our final example, we will iterate Theorem 2.2 on the b^k -tangent bundle. For ease, we fix a Z defining function so as to avoid needing to specifying jet data in our description.

Example 2.7. Given a pair (M, Z) , the idea of the b^k -tangent bundle is to be the vector bundle whose sections are all tangent to Z and have order k degeneracy in the direction normal to Z . If x is a defining function for Z , and y_1, \dots, y_n are local coordinates in Z , then the vector fields

$$x^k \frac{\partial}{\partial x}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}$$

form a local basis for b^kTM .

By iteratively applying Theorem 2.2 to the b^k -tangent bundle $b^kTM \rightarrow M$ and rescaling by the subbundle $0 \rightarrow Z$, the resulting bundle ${}^{0^m}(b^kTM) \rightarrow M$ is called the $(0^m; b^k)$ -***tangent bundle***. The vector fields

$$x^{k+m} \frac{\partial}{\partial x}, x^m \frac{\partial}{\partial y_1}, \dots, x^m \frac{\partial}{\partial y_n}$$

form a local basis for ${}^{0^m}(b^kTM)$. The dual bundle ${}^{0^m}(b^kT^*M) \rightarrow M$ is called the $(0^m; b^k)$ -***cotangent bundle*** and is locally generated by

$$\frac{dx}{x^{k+m}}, \frac{dy_1}{x^m}, \dots, \frac{dy_n}{x^m}.$$

Note that this construction is not obtained from the rescaling construction without more data, e.g., a tubular neighborhood decomposition of M near Z . In fact, it is easy to check that a different choice x of Z defining function can produce a different bundle.

2.2. Differential forms on \mathcal{A} -manifolds. To every algebroid there is an associated cohomology theory. Recall, for any Lie algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ over M , with dual \mathcal{A}^* , the degree k \mathcal{A} -forms are

$$\mathcal{A}\Omega^k(M) = \Gamma(\wedge^k \mathcal{A}^*),$$

the sections of the k th exterior power of the dual bundle \mathcal{A}^* . The differential operator $d_{\mathcal{A}}$ acting on ${}^{\mathcal{A}}\Omega^*(M)$, $d_{\mathcal{A}} : {}^{\mathcal{A}}\Omega^k(M) \rightarrow {}^{\mathcal{A}}\Omega^{k+1}(M)$ is defined by

$$\begin{aligned} (d_{\mathcal{A}}\beta)(\alpha_0, \alpha_1, \dots, \alpha_k) &= \sum_{i=0}^k (-1)^i \rho_{\mathcal{A}}(\alpha_i) \cdot \beta(\alpha_0, \dots, \hat{\alpha}_i, \dots, \alpha_k) \\ &+ \sum_{0 \leq i < j \leq k} (-1)^{i+j} \beta([\alpha_i, \alpha_j]_{\mathcal{A}}, \alpha_0, \dots, \hat{\alpha}_i, \dots, \hat{\alpha}_j, \dots, \alpha_k) \end{aligned}$$

for $\beta \in {}^{\mathcal{A}}\Omega^k(M)$, and $\alpha_0, \dots, \alpha_k \in \Gamma(\mathcal{A})$. This is a complex whose cohomology is called the Lie algebroid cohomology or \mathcal{A} -de Rham cohomology.

2.2.1. The Taylor series of an ${}^F\mathcal{A}$ -form. Developing a notion of Taylor series for ${}^F\mathcal{A}$ -forms, forms of an algebroid \mathcal{A} rescaled using a subbundle $F \subseteq \mathcal{A}|_Z \rightarrow Z$, is a useful tool in computing and understanding algebroid cohomology. We will discuss three notions of Taylor expansion. Definition 2.8 is valuable because it is invariant under change of Z defining function. Definition 2.9 does depend on Z defining function, but is more convenient for computations.

Definition 2.8. Let $Z \subset M$ and let \mathcal{A} be an algebroid over (M, Z) . Consider ${}^F\mathcal{A}$, the algebroid formed by rescaling \mathcal{A} using a subbundle $F \subseteq \mathcal{A}|_Z \rightarrow Z$. Let $J_Z \subseteq \mathcal{C}^\infty(M)$ be the ideal of functions that vanish at Z . The k -jet of a section $\sigma \in {}^F\mathcal{A}\Omega^p(M)$ at Z will be the projection of σ onto

$${}^{\mathcal{A}}\Omega^p(M) / ((J_Z)^{k+1} \cdot {}^{\mathcal{A}}\Omega^p(M)).$$

In practice we fix a Z defining function x and τ a tubular neighborhood decomposition of M near Z and work with the k -jet as a Taylor series expansion of order k . We take a connection ∇ on ${}^F\mathcal{A}$ and we fix a vector field N transverse to Z and a Z defining function x such that $N(x) = 1$.

Definition 2.9. The *Taylor series* of order k associated to this data and a section $\sigma \in {}^F\mathcal{A}\Omega^p(M)$ is the sum

$$\sum_{i=0}^k x^i \sigma_i \quad \text{where} \quad \sigma_i = \frac{1}{i!} (\nabla_N)^i \sigma|_Z$$

and $(\sigma_0, \sigma_1, \dots, \sigma_k)$ represents the k -jet of the section at Z in that

$$\sigma - \chi \left(\sum_{i=0}^k x^i \sigma_i \right) \in J_Z^{k+1}$$

for any cut-off function χ supported in τ with $\chi \equiv 1$ near Z .

By an abuse of notation, when we have rescaled an algebroid often it is convenient to think of forms in ${}^F\mathcal{A}$ near Z as forms in \mathcal{A} with singular coefficients:

Remark 2.10. Given any Z defining function x , every ${}^F\mathcal{A}$ -form will admit a Taylor series expansion in x . If v_1^*, \dots, v_k^* is a local basis of F^* at Z , we can extend to a local smooth basis $v_1^*, \dots, v_k^*, v_{k+1}^*, \dots, v_a^*$ of \mathcal{A}^* in a subset U of a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$. Then $v_1^*, \dots, v_k^*, \frac{v_{k+1}^*}{x}, \dots, \frac{v_a^*}{x}$ is a local basis of ${}^F\mathcal{A}^*$ in τ . Then a degree- p ${}^F\mathcal{A}$ form σ locally is

$$\sum_{i_1, i_2, \dots, i_p=1}^M f_{i_1 i_2 \dots i_p} \frac{v_{i_1}^*}{x^{\delta_I(i_1)}} \wedge \dots \wedge \frac{v_{i_p}^*}{x^{\delta_I(i_p)}}$$

where δ_I is the indicator function of the set $I = \{k+1, \dots, a\}$, and for a collection of smooth functions $f_{i_1 i_2 \dots i_p} \in \mathcal{C}^\infty(U)$. By taking the Taylor series expansion of $f_{i_1 i_2 \dots i_p}$ in x , $f_{i_1 i_2 \dots i_p} = x^{k_{i_1 i_2 \dots i_p}} \tilde{f}_{i_1 i_2 \dots i_p}$ where $k_{i_1 i_2 \dots i_p}$ is the maximal possible such non-negative integer. Then locally

$$\sigma = \sum_{i_1, i_2, \dots, i_p=1}^M \frac{x^{k_{i_1 i_2 \dots i_p}}}{x^{\delta_I(i_1) + \dots + \delta_I(i_p)}} \tilde{f}_{i_1 i_2 \dots i_p} v_{i_1}^* \wedge \dots \wedge v_{i_p}^*,$$

a Taylor series expansion in x on U . Because Z is compact, we can use a partition of unity to express σ in this form on the tubular neighborhood τ of Z .

Now we are equipped to compute the Lie algebroid cohomology of the scattering tangent bundle. Note that in our examples, since the Lie bracket is inherited from the usual Lie bracket, we also have that the \mathcal{A} -differential is computable by using the de Rham differential on $M \setminus Z$. To motivate our expression for the scattering de-Rham cohomology, we offer a brief discussion of the role played by a Z defining function.

2.2.2. Z defining functions and density bundles. The Lie algebroids \mathcal{A} obtained from rescaling the tangent bundle, e.g. ${}^bTM, {}^0TM, {}^{sc}TM$, but not ${}^{b^k}TM$, do not depend on a choice of Z defining function, hence neither do their \mathcal{A} -de Rham cohomologies. However, it is convenient for computations to work in a fixed tubular neighborhood, and so with a fixed Z defining function.

Example 2.11. Let x be a Z defining function on a manifold (M, Z) . Then $v \in {}^b\Omega^k(M)$ can be expressed as $v = \frac{dx}{x} \wedge \alpha + \beta$ for $\alpha, \beta \in \Omega^*(M)$. Then, as in Remark 2.10, $dv = \frac{dx}{x} \wedge (x\partial_x\beta - d\alpha) + d\beta$ is a b -form with d the usual differential applied on $M \setminus Z$. Notice that $dv = \frac{dx}{x} \wedge (x\partial_x\beta - d\alpha) + d\beta = 0$ if and only if

$$d\alpha = x\partial_x\beta \text{ and } d\beta = 0.$$

We can express any other Z defining function \tilde{x} as $x = f\tilde{x}$ for some nowhere vanishing function $f \in \mathcal{C}^\infty(M)$. Then

$$\frac{dx}{x} = \frac{\tilde{x}df + f d\tilde{x}}{\tilde{x}f} = \frac{df}{f} + \frac{d\tilde{x}}{\tilde{x}} \quad \text{and} \quad \frac{dx}{x} \wedge \alpha + \beta = \frac{df}{f} \wedge \alpha + \frac{d\tilde{x}}{\tilde{x}} \wedge \alpha + \beta.$$

Notice that $\log|f|$ is a smooth function because f is non-vanishing. Then $d(\log|f| \wedge \alpha) = \frac{df}{f} \wedge \alpha$ because α is closed. Thus, in cohomology

$$\left[\frac{dx}{x} \wedge \alpha + \beta \right] = \left[\frac{d\tilde{x}}{\tilde{x}} \wedge \alpha + \beta \right]$$

and the cohomology class is unambiguous despite a representative being expressed using a particular Z defining function.

The change of Z defining function for a scattering cohomology representative, on the other hand, is not canonically trivial in this way.

Example 2.12. Let x be a Z defining function on a manifold (M, Z) . Consider the scattering-form $v = \frac{dx}{x^{k+1}} \wedge \alpha + \frac{\beta}{x^k}$ for $\alpha, \beta \in \Omega^*(M)$. As above, we express any other Z defining function \tilde{x} as $x = f\tilde{x}$ for some nowhere vanishing positive function $f \in \mathcal{C}^\infty(M)$. Then

$$\frac{dx}{x^{k+1}} \wedge \alpha + \frac{\beta}{x^k} = \frac{d\tilde{x}}{\tilde{x}^{k+1}} \wedge \left(\frac{1}{f^k} + \frac{\partial_{\tilde{x}} f}{f^{k+1}} \tilde{x} \right) \alpha + \frac{1}{\tilde{x}^k} \left(\frac{d_Z f}{f^{k+1}} \wedge \alpha + \frac{\beta}{f^k} \right).$$

Thus the α and β decomposition is highly dependent on the choice of Z defining function. We will show through the course of proving Theorem 2.15 that the only real ambiguity above is the scaling of α by $\frac{1}{f^k}$. This apparent dependence on x is accounted for by the density bundle in equation (2.3), the restriction of a rescaled bundle to Z . A more complete discussion of densities can be found in section 4.5 of [21].

Definition 2.13. Let $p \in Z$. The space of s -densities on N_p^*Z is, for $s \in \mathbb{R}$, the space

$$|N_p^*Z|^s = \left\{ \psi : N_p^*Z \setminus \{0\} \rightarrow \mathbb{R} \mid \psi(\lambda V) = |\lambda|^s \psi(V) \quad \forall V \in N_p^*Z \setminus \{0\}, \lambda \neq 0 \right\}$$

and the s -densities on N^*Z over Z is the bundle

$$|N^*Z|^s = \bigsqcup_{p \in Z} |N_p^*Z|^s.$$

A Z defining function x induces a trivialization $|dx|^s$ of $|N^*Z|^s \rightarrow Z$. If $\tilde{x} = fx$ is another Z defining function, then $|d\tilde{x}|^s = f^s |dx|^s$ as sections of $|N^*Z|^s$.

Because ${}^{sc}TM|_Z \simeq (TZ \oplus (N^*Z \otimes TM|_Z/TZ)) \otimes N^*Z$, a density bundle appears in the cohomology groups to account for changes of Z defining function and our presentation of the cohomology is independent of Z defining function.

2.2.3. Scattering de-Rham cohomology. Our computation of scattering de-Rham cohomology utilizes the following result of Rafe Mazzeo and Richard Melrose ([21], Prop. 2.49).

Theorem 2.14 (Mazzeo-Melrose). *Let bTM be the b -tangent bundle associated to (M, Z) . The b -de Rham cohomology is*

$${}^bH^p(M) \simeq H^p(M) \oplus H^{p-1}(Z).$$

Theorem 2.15. *Let (M, Z) be a manifold M with hypersurface $Z \subset M$. Then ${}^{sc}H^p(M)$, the Lie algebroid cohomology of the scattering tangent bundle ${}^{sc}TM$ over (M, Z) , is isomorphic to*

$${}^bH^p(M) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p}) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p}).$$

Proof. The bundle map $i : {}^{sc}TM \rightarrow {}^bTM$ constructed in Theorem 2.2 is an inclusion of Lie algebroids and hence fits into a short exact sequence of complexes

$$0 \rightarrow {}^b\Omega^p(M) \xrightarrow{i^*} {}^{sc}\Omega^p(M) \xrightarrow{\pi} \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = {}^{sc}\Omega^p(M)/{}^b\Omega^p(M).$$

The differential on $\mathcal{C}d$ is induced by the differential ${}^{sc}d$ on ${}^{sc}\Omega^p(M)$: if π is the projection ${}^{sc}\Omega^p(M) \rightarrow {}^{sc}\Omega^p(M)/{}^b\Omega^p(M)$, then $\mathcal{C}d(\eta) = \pi({}^{sc}d(\theta))$ where $\theta \in {}^{sc}\Omega^p(M)$ is any form such that $\pi(\theta) = \eta$. Hence $(\mathcal{C}d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}d)$ is in fact a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , note that x defines a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$ of N^*Z . We can write a degree p scattering form $\nu \in {}^{sc}\Omega^p(M)$ as

$$\nu = \theta + \sum_{i=0}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i + \frac{\beta_i x^i}{x^p} \right)$$

where $\theta \in {}^b\Omega^p(M)$, and $\alpha_i, \beta_i \in \Omega^*(Z) \simeq \Omega^*(Z; |N^*Z|^{-p})$ by $(t_x)_*$.

We write $\mathcal{R}_b(\nu) = \theta$ and $\mathcal{S}_b(\nu) = \nu - \mathcal{R}_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $\mathcal{R}_b({}^{sc}d\nu) = {}^{sc}d(\mathcal{R}_b(\nu))$ and $\mathcal{S}_b({}^{sc}d\nu) = {}^{sc}d(\mathcal{S}_b(\nu))$. Thus the trivialization τ induces a splitting ${}^{sc}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence ${}^{sc}H^p(M) = {}^bH^p(M) \oplus H^p(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

Because

$$\pi({}^{sc}d\nu) = \frac{1}{x^{p+1}} \left(\sum_{i=0}^{p-1} \frac{dx}{x} \wedge (x^{i+1}(-d\alpha_i - (p-i)\beta_i)) + x^{i+1}d\beta_i \right)$$

then $\pi({}^{sc}d\nu) = 0$ if and only if $\beta_i = \frac{-d\alpha_i}{(p-i)}$ for all $i = 0, \dots, p-1$. Thus

$$\ker({}^{\mathcal{C}}d : \mathcal{C}^p \rightarrow \mathcal{C}^{p+1}) = \left\{ \sum_{i=0}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i - \frac{d\alpha_i x^i}{(p-i)x^p} \right) \middle| \alpha_i \in \Omega^{p-1}(Z) \right\}.$$

Now we will consider the image of ${}^{\mathcal{C}}d : \mathcal{C}^{p-1} \rightarrow \mathcal{C}^p$. There exists

$$\tilde{\nu} = \sum_{i=1}^{p-1} \frac{\alpha_i x^{i-1}}{-(p-i)x^{p-1}} \in \mathcal{C}^{p-1} \text{ such that } {}^{\mathcal{C}}d\tilde{\nu} = \sum_{i=1}^{p-1} \left(\frac{dx}{x^{p+1}} \wedge \alpha_i x^i - \frac{d\alpha_i x^i}{(p-i)x^p} \right),$$

and hence

$$\nu - d\tilde{\nu} = \frac{dx}{x^{p+1}} \wedge \alpha_0 - \frac{d\alpha_0}{px^p}.$$

Since

$$\text{Im}({}^{\mathcal{C}}d) \subseteq \{\alpha_0 = 0\},$$

this shows that each such form represents a distinct cohomology class. Thus we have identified $H^p(\mathcal{C}^*)$.

Next, consider the effect of choice of Z defining function. We will show that we can identify

$$H^p((\mathcal{C}^*, {}^{\mathcal{C}}d)) \simeq \Omega^{p-1}(Z; |N^*Z|^{-p}) \text{ by identifying } \frac{dx}{x^{p+1}} \wedge \alpha_0 - \frac{d\alpha_0}{px^p} \text{ with } \alpha_0.$$

Indeed, each choice of x defines a trivialization of N^*Z , $t_x : N^*Z \rightarrow \mathbb{R}$. Then $(t_x)_*$ gives an isomorphism $\Omega^{p-1}(Z) \simeq \Omega^{p-1}(Z; |N^*Z|^{-p})$. To see that this is well-defined, note that changing x to another Z defining function \tilde{x} , means that $\tilde{x} = \phi x$ for some non-vanishing function $\phi \in \mathcal{C}^\infty(M)$. Then $\frac{d\tilde{x}}{\tilde{x}} = \frac{dx}{x} + \frac{d\phi}{\phi}$.

Since ϕ is a positive function,

$$\left[\frac{d\tilde{x}}{\tilde{x}^{p+1}} \wedge \alpha_0 \right] \text{ and } \left[\frac{dx}{x^{p+1}} \wedge \frac{\alpha_0}{\phi^p} \right]$$

are representatives of the same cohomology class in $H^p(\mathcal{C})$ and

$$|d\tilde{x}|^{-p} = \phi^{-p} |dx|^{-p}$$

gives the change of trivialization of the density bundle $|N^*Z|^{-p}$. Hence, the cohomology group at \mathcal{C}^p is $\Omega^{p-1}(Z; |N^*Z|^{-p})$, smooth $p-1$ forms on Z valued in $|N^*Z|^{-p}$.

We have shown that ${}^{sc}H^p(M) \simeq {}^bH^p(M) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p})$. The final isomorphism is a consequence of ${}^bH^p(M) \simeq H^p(M) \oplus H^{p-1}(Z)$. \square

2.3. \mathcal{A} -symplectic structures. We can use algebroids to define generalized symplectic geometries and use their respective de Rham cohomologies to establish analogues of certain standard results in symplectic geometry.

Definition 2.16. Given a rank $2n$ Lie algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ over a manifold M , an \mathcal{A} -symplectic structure on M is a closed, non-degenerate, degree-2 \mathcal{A} -form $\omega_{\mathcal{A}} \in {}^{\mathcal{A}}\Omega^2(M)$. That is ${}^{\mathcal{A}}d\omega_{\mathcal{A}} = 0$ and $\wedge^n(\omega_{\mathcal{A}}) \neq 0$.

Given a Lie algebroid $(\mathcal{A}', [\cdot, \cdot]_{\mathcal{A}'}, \rho_{\mathcal{A}'})$ over a manifold M , a Poisson manifold (M, π) is \mathcal{A}' -Poisson if there exists an \mathcal{A}' bi-vector $\pi_{\mathcal{A}'} \in \Gamma(\wedge^2 \mathcal{A}')$ that realizes π , i.e. $\rho_{\mathcal{A}'}(\pi_{\mathcal{A}'}) = \pi$.

If rank \mathcal{A}' is even, and $\pi_{\mathcal{A}'}$ is non-degenerate, then we recover the correspondence analogous to (M, ω) symplectic and (M, π) non-degenerate Poisson: there is a one-to-one correspondence between \mathcal{A} -symplectic forms $\omega_{\mathcal{A}}$ and non-degenerate \mathcal{A} -Poisson bi-vectors $\pi_{\mathcal{A}}$.

$$\mathcal{A} \begin{array}{c} \xrightarrow{\omega_{\mathcal{A}}^b} \\ \xleftarrow{\pi_{\mathcal{A}}^{\sharp} = (\omega_{\mathcal{A}}^b)^{-1}} \end{array} \mathcal{A}^*$$

Remark 2.17. Note that while we allow degenerate \mathcal{A} -Poisson structures, in [12] b -Poisson bi-vectors are necessarily non-degenerate.

2.3.1. \mathcal{A} -Moser. Given an algebroid $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ over a manifold (M, Z) such that the anchor map ρ is an isomorphism off of the hypersurface Z , we can employ the smooth Moser trick to recover certain analogues of standard neighborhood theorems from symplectic geometry.

Definition 2.18. A map $\phi : (M_1, Z_1) \rightarrow (M_2, Z_2)$ is an \mathcal{A} -map if $\phi^* : {}^{\mathcal{A}}\Omega^1(M_2)|_{M_2 \setminus Z_2} \rightarrow {}^{\mathcal{A}}\Omega^1(M_1)|_{M_1 \setminus Z_1}$ extends to map ϕ^* on all forms in \mathcal{A} .

Recall from [12] that a b -map is a map $f : (M_1, Z_1) \rightarrow (M_2, Z_2)$ transverse to Z_2 and such that $f^{-1}(Z_2) = Z_1$. In the examples of algebroids \mathcal{A} that we have provided, to be an \mathcal{A} map it suffices to be a b -map.

Definition 2.19. Given two \mathcal{A} -symplectic forms ω_1, ω_2 on (M, Z) , an \mathcal{A} -symplectomorphism is an \mathcal{A} -map $\phi : M \rightarrow M$ such that $\phi^*\omega_2 = \omega_1$.

Note since $\rho_{\mathcal{A}}$ is an isomorphism of \mathcal{A} and TM away from Z , there is an inverse map $\rho_{\mathcal{A}}^{-1} : TM|_{M \setminus Z} \rightarrow \mathcal{A}|_{M \setminus Z}$. Thus, away from Z , we can pull back forms on \mathcal{A} to forms on TM .

Lemma 2.20. Let $(\mathcal{A}, [\cdot, \cdot]_{\mathcal{A}}, \rho_{\mathcal{A}})$ be an algebroid over (M^{2n}, Z) such that $\rho_{\mathcal{A}}$ is an isomorphism on $M \setminus Z$. Let $\omega_1, \omega_2 \in {}^{\mathcal{A}}\Omega^2(M)$ satisfy

- (1) $d_{\mathcal{A}}(\omega_1) = d_{\mathcal{A}}(\omega_2) = 0$,
- (2) $(\rho_{\mathcal{A}}^{-1})^*(\omega_2 - \omega_1) \in \Omega^*(M \setminus Z)$ extends smoothly to 0 on Z , and

(3) $\wedge^n \omega_i(p) \neq 0$ for all $p \in Z$.

Then there exist neighborhoods U_1, U_2 of Z and a diffeomorphism $\phi : U_1 \rightarrow U_2$ such that $\phi^*(\omega_2|_{U_2}) = \omega_1|_{U_1}$ and $\phi|_Z$ is the identity map on Z .

Proof. We choose a small enough neighborhood $U_0 \supset Z$ such that ω_1 and ω_2 are non-degenerate, and thus are \mathcal{A} -symplectic, on U_0 . Perhaps taking a smaller neighborhood if necessary, we have that

$$\omega_t = \omega_1 + t(\omega_2 - \omega_1)$$

is a family of \mathcal{A} -symplectic forms. In following the standard Moser's argument, for instance see [2] Ch.7, we will show that there is an \mathcal{A} -one form σ such that

$$\frac{d\omega_t}{dt} = \omega_2 - \omega_1 = d\sigma.$$

Note that the smooth family of forms $(\rho_{\mathcal{A}}^{-1})^*(\omega_2 - \omega_1)$ on $M \setminus Z$ with smooth extension to 0 on Z is exact on U_0 . Thus, by the standard Moser trick on smooth forms,

$$(\rho_{\mathcal{A}}^{-1})^*(\omega_2 - \omega_1) = d\tilde{\sigma}$$

for some smooth form $\tilde{\sigma}$ and we can integrate to obtain the desired smooth isotopy. \square

As a consequence of this lemma, to establish an \mathcal{A} -Darboux theorem, it suffices to show what an \mathcal{A} -symplectic form must look like on a small neighborhood of the hypersurface Z .

2.3.2. \mathcal{A} -symplectic realization. Interestingly, some geometrically natural algebroids, such as the zero tangent bundle 0TM , do not admit symplectic structures on manifolds of dimension greater than 2. The ${}^{0^m}({}^{b^k}TM)$ bundles defined in Example 2.7 are another example of Algebroids that fail to admit a symplectic form. These examples all satisfy a certain fiber description: let the fibers of an algebroid \mathcal{A} over (M, Z) and its dual \mathcal{A}^* satisfy

$$\mathcal{A}_p \simeq T_p M \text{ if } p \notin Z \text{ and } \mathcal{A}_p \simeq x^m(T_p Z) + \langle x^{k+m} \frac{\partial}{\partial x} \rangle \text{ if } p \in Z \quad (2.4)$$

$$\mathcal{A}^*_p \simeq T_p^* M \text{ if } p \notin Z \text{ and } \mathcal{A}^*_p \simeq \frac{1}{x^m}(T_p^* Z) + \langle \frac{dx}{x^{k+m}} \rangle \text{ if } p \in Z$$

where x is a defining function for Z .

Proposition 2.21. *Let M be a manifold of dimension greater than 2 and Z any non-empty hypersurface $Z \subset M$. Let \mathcal{A} be an algebroid over (M, Z) satisfying (2.4). If $m > 0$, and $k \neq 1$, then M does not admit an \mathcal{A} -symplectic structure.*

Proof. An \mathcal{A} -symplectic form $\omega_{\mathcal{A}} \in {}^{\mathcal{A}}\Omega^2(M)$ would be expressible in a tubular neighborhood of Z as

$$\omega_{\mathcal{A}} = \frac{dx}{x^{k+m}} \wedge \alpha + \frac{\beta}{x^m}$$

for a smooth 1-form α and smooth 2-form β on M . Consider

$$d\omega_{\mathcal{A}} = -\frac{dx}{x^{k+m}} \wedge d\alpha + \frac{d\beta}{x^m} - m\frac{dx}{x^{m+1}} \wedge \beta.$$

Because $k+m \neq m+1$, closedness of $\omega_{\mathcal{A}}$ provides the relation $\beta = 0$. However, $\omega_{\mathcal{A}}$ non-degenerate means

$$\wedge^n(\omega_{\mathcal{A}}) = \frac{dx}{x^{(k+mn)}} \wedge \alpha \wedge \beta^{n-1} \neq 0.$$

Thus manifolds of dimension greater than 2 do not admit degree 2, non-degenerate, closed \mathcal{A} -forms for algebroids of the specified form. \square

3. Scattering Symplectic Geometry

As we noted in the introduction, scattering-symplectic geometry includes the study of the standard Euclidean symplectic form at infinity. We also observed that the standard form on \mathbb{R}^{2n} extends to a scattering-symplectic form that induces a contact structure on the boundary sphere at infinity. Contact structures arising in this way are imposed by all scattering-symplectic structures. As a hypersurface in an \mathcal{A} -manifold, the existence of a Z defining function x means Z is co-orientable. The scattering-symplectic structure imposes the further restriction that Z is also co-orientable as a contact manifold.

Proposition 3.1. *If (M, Z, ω) is a scattering-symplectic manifold, then ω induces a co-oriented contact structure ξ on Z .*

Proof. Let x be a Z defining function. A scattering 2-form ω can be expressed near Z as

$$\omega = \frac{dx}{x^3} \wedge \alpha + \frac{\beta}{x^2}$$

for some smooth forms $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$. Because ω is closed, $\beta = \frac{-d\alpha}{2}$ at Z . Further, because ω is a non-degenerate scattering 2-form, we know that

$$\frac{dx}{x^{2n+1}} \wedge \alpha \wedge \left(\frac{-d\alpha}{2} \right)^{n-1} \neq 0$$

at Z . Thus $\alpha \wedge (d\alpha)^{n-1} \neq 0$ as a smooth form on Z . If we express ω using a different Z defining function \tilde{x} such that $\phi\tilde{x} = x$ for some non-vanishing function $\phi \in \mathcal{C}^\infty(M)$, then

$$\omega|_Z = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2} = \frac{1}{(\phi\tilde{x})^2} \left(\frac{d\phi}{\phi} + \frac{d\tilde{x}}{\tilde{x}} \right) \wedge \alpha - \frac{d\alpha}{2(\phi\tilde{x})^2} = \frac{d\tilde{x}}{\tilde{x}^3} \wedge \left(\frac{\alpha}{\phi^2} \right) - \frac{d(\alpha/\phi^2)}{2\tilde{x}^2}.$$

The contact form $\tilde{\alpha}$ induced by ω expressed in \tilde{x} satisfies $\tilde{\alpha} = \frac{\alpha}{\phi^2}$, and thus is conformally equivalent to α . Thus the scattering symplectic form ω induces a conformal class of contact forms defining the contact structure $\ker \alpha = \xi$ on Z . \square

We will explore the relationship between a contact hypersurface and a scattering-symplectic manifold in Section 4. The existence of an induced contact structure evidences the fact that scattering-symplectic structures are sufficiently rigid to all locally look the same.

Proposition 3.2. *(a sc-Darboux theorem) Let ω be a sc-symplectic form on (M, Z) and let $p \in Z$. There exists a coordinate chart $(U, x_1, y_1, \dots, x_n, y_n)$ centered at p such that on U , the hypersurface Z is locally defined by $\{x_1 = 0\}$, and*

$$\omega = \frac{dx_1}{x_1^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x_1^2}.$$

Proof. Let ω be a scattering symplectic form on (M, Z) . By Proposition 3.1, given a Z defining function x ,

$$\omega|_Z = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}$$

where α is a contact form on Z . Let $p \in Z$. In a neighborhood $U_p \subset Z$, there exist contact-Darboux coordinates $y_1, x_2, y_2, \dots, x_n, y_n$ such that

$$\alpha = dy_1 + \sum_{i=2}^n (y_i dx_i - x_i dy_i).$$

Then

$$\omega|_Z = \frac{dx}{x^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x^2}.$$

In the set $U_p \times \{|x| < \varepsilon\}$, choose

$$\omega_0 = \frac{dx}{x^3} \wedge \left(dy_1 + \sum_{i=2}^n y_i dx_i - x_i dy_i \right) + \sum_{i=2}^n \frac{dx_i \wedge dy_i}{x^2}.$$

Note that $(\rho_{sc}^{-1})^*(\omega - \omega_0)$ is a ‘standard’ differential form that is closed on a contractible set, so by the usual Poincaré lemma, $(\rho_{sc}^{-1})^*(\omega - \omega_0) = d\nu$ for some smooth form ν on $U_p \times \{|x| < \varepsilon\}$. By using a homotopy operator, we can assume $\nu|_Z = 0$ and we have a smooth extension of $(\rho_{sc}^{-1})^*(\omega - \omega_0)$ to 0 on Z . By the \mathcal{A} -Moser Lemma 2.20, we have the desired result. \square

We can further employ the \mathcal{A} -Moser lemma to establish a tubular neighborhood theorem for ω near Z .

Recall from Theorem 2.15 that

$${}^{sc}H^2(M) \simeq H^2(M) \oplus H^1(Z) \oplus \Omega^2(Z; |N^*Z|^{-2}).$$

Given a cohomology class $[\sigma] \in {}^{sc}H^2(M)$, we can associate to it a *decomposition* (a, b_1, b_2) where $a \in \Omega^2(Z; |N^*Z|^{-2})$, $b_1 \in H^1(Z)$, and $b_2 \in H^2(M)$. We will consider scattering symplectic forms ω and their cohomology decompositions (a, b_1, b_2) . A given Z defining function x gives us a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$ and defines a smooth contact form $\alpha = (t_x)_*(a) \in \Omega^1(Z)$. We will show that for any $\beta_i \in b_i$, there is a tubular neighborhood of Z such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2. \quad (3.1)$$

Proposition 3.3. *Let (M, Z, ω) be a scattering symplectic manifold. Given a Z defining function x , there exists a tubular neighborhood $U \supset Z$, a contact form α , and closed forms $\beta_1 \in \Omega^1(Z)$, $\beta_2 \in \Omega^2(Z)$ such that on U there exists a scattering-symplectomorphism pulling ω back to (3.1).*

Proof. Let ω be a scattering symplectic form on a manifold (M, Z) with cohomology class decomposition (a, b_1, b_2) . Let x be a Z defining function and $t_x : N^*Z \rightarrow \mathbb{R}$ the associated trivialization. Let α denote $(t_x)_*(a)$ and let U be a tubular neighborhood of Z . Choose a closed form β_2 that is cohomologous to $b_2|_U$ and choose a representative $\beta_1 \in b_1$. Let

$$\omega_0 = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

Then $\omega|_U - \omega_0 = d\nu$. By using a homotopy operator, we can assume $\nu|_Z = 0$ and we have a smooth extension of $(\rho_{sc}^{-1})^*(\omega|_U - \omega_0)$ to 0 on Z . By the \mathcal{A} -Moser Lemma 2.20, we have the desired result. \square

In the previous two results, we wrote a scattering-symplectic form in a standard way by assuming that ω fixed a contact structure on Z . In general a scattering-symplectomorphism will induce a contactomorphism rather than merely fix the contact structure.

Proposition 3.4. *If there exists a scattering-symplectomorphism*

$$\Phi : (M_1, Z_1, \omega_1) \rightarrow (M_2, Z_2, \omega_2),$$

then $\Phi|_{Z_1} : Z_1 \rightarrow Z_2$ is a contactomorphism between the contact structures induced by ω_1 and ω_2 respectively.

Proof. Given a Z_2 defining function x_2 , we can write

$$\omega_2|_{Z_2} = \frac{dx_2}{x_2^3} \wedge \alpha_2 - \frac{1}{2} \frac{d\alpha_2}{x_2^2}$$

for some contact form α_2 on Z_2 . Then

$$\Phi^*\omega_2|_{Z_2} = \frac{d(\Phi^*x_2) \wedge \Phi^*\alpha_2}{(\Phi^*x_2)^3} - \frac{1}{2} \frac{\Phi^*d\alpha_2}{(\Phi^*x_2)^2} = \frac{dx_1}{x_1^3} \wedge \alpha_1 - \frac{1}{2} \frac{d\alpha_1}{x_1^2} = \omega_1|_{Z_1}$$

for some Z_1 defining function x_1 and contact form α_1 . We will compare the terms in this equality. Note since Φ preserves the singular locus of ω_1 and ω_2 , then $\Phi^*x_2 = fx_1$ for positive $f \in \mathcal{C}^\infty(M_1)$. So

$$d(\Phi^*x_2) = d(fx_1) = \frac{x_1df + fdx_1}{x_1^3f^3} \quad \text{and} \quad \frac{x_1df + fdx_1}{x_1^3f^3} \Big|_Z = \frac{dx_1}{x_1^3f^2}.$$

Then $\frac{dx_1}{x_1^3f^2} \wedge \Phi^*\alpha_2 = \frac{dx_1}{x_1^3} \wedge \alpha_1$ and thus $\Phi^*\alpha_2 = f^2\alpha_1$. \square

In fact, we can use certain contactomorphisms to construct local scattering-symplectomorphisms.

Proposition 3.5. *Let ω and $\tilde{\omega}$ be scattering-symplectic forms on (M, Z) with cohomology decompositions (a, b_1, b_2) for $[\omega]$ and $(\tilde{a}, \tilde{b}_1, \tilde{b}_2)$ for $[\tilde{\omega}]$. Let x be a Z defining function and consider the induced trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. If there is a contactomorphism $\Phi : Z \rightarrow Z$ such that*

- (1) $\Phi^*(t_x)_*a = f \cdot (t_x)_*\tilde{a}$ for positive $f \in C^\infty(Z)$,
- (2) $\Phi^*b_1 = \tilde{b}_1$, and
- (3) $\Phi^*b_2|_Z = \tilde{b}_2|_Z$,

then there exists a tubular neighborhood $U \supset Z$ and a scattering-symplectomorphism $\phi : U \rightarrow U$ such that $\phi^\omega_1 = \omega_2$.*

Proof. Fix a Z defining function x and consider the induced trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. Let $\alpha = \Phi^*(t_x)_*a$ and $\tilde{\alpha} = (t_x)_*\tilde{a}$. Let $\beta_1 \in b_1$ and $\beta_2 \in b_2$. By Proposition 3.3, there exists a tubular neighborhood $Z \times \{x \in (-\delta, \delta)\}$ on which

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

Let $\Phi : Z \rightarrow Z$ be a contactomorphism such that $\Phi^*(\alpha) = e^g\tilde{\alpha}$, for a smooth function $g \in \mathcal{C}^\infty(Z)$. Define a function

$$f : Z \times (-\delta, \delta) \rightarrow \mathbb{R} \quad \text{by} \quad f(z, x) = \sqrt{e^{g(z)}}x.$$

Then consider

$$\Psi : Z \times (-\delta, \delta) \rightarrow Z \times (-\delta, \delta) \quad \text{given by} \quad \Psi(z, x) = (\Phi(z), f(z, x)).$$

Since $\frac{d(e^{g/2}x)}{e^{3g/2}x^3} = \frac{dg}{2e^gx^2} + \frac{dx}{e^gx^3}$, we have that

$$\Psi^*(\omega) = \frac{dx}{x^3} \wedge \tilde{\alpha} + \frac{dx}{x} \wedge \Phi^*\beta_1 - \frac{d\Phi^*\alpha}{2e^gx^2} + \Phi^*\beta_2 + \frac{dg}{2x^2} \wedge \tilde{\alpha} + \frac{dg}{2} \wedge \Phi^*\beta_1.$$

Because $-\frac{d\Phi^*\alpha}{2e^g x^2} = -\frac{e^g dg}{2e^g x^2} \wedge \tilde{\alpha} - \frac{e^g d\tilde{\alpha}}{2e^g x^2}$, we have that

$$\Psi^*(\omega) = \frac{dx}{x^3} \wedge \tilde{\alpha} - \frac{d\tilde{\alpha}}{2x^2} + \frac{dx}{x} \wedge \Phi^*\beta_1 + \Phi^*\beta_2 + \frac{dg}{2} \wedge \Phi^*\beta_1.$$

Note that $\Phi^*\beta_1$ is closed, so $d\left(\frac{g}{2}\Phi^*\beta_1\right) = \frac{dg}{2} \wedge \Phi^*\beta_1$ is an exact form. Since $\Phi^*\tilde{b}_1 = b_1$ and $\Phi^*\tilde{b}_2|_Z = b_2|_Z$, we have $\Phi^*\beta_1 \in \tilde{b}_1$ and $\Phi^*\beta_2 + \frac{dg}{2} \wedge \Phi^*\beta_1 \in \tilde{b}_2$. Thus by Proposition 3.3, there exists a scattering-symplectomorphism between $\Psi^*\omega$ and $\tilde{\omega}$ on a tubular neighborhood of Z . \square

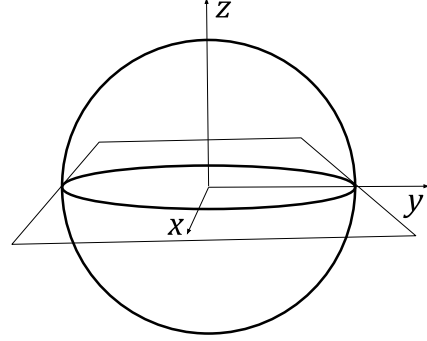
3.1. Scattering-Symplectic Spheres. We will conclude this section by providing an example of scattering-symplectic manifolds. All even dimensional spheres admit scattering-symplectic structures.

Let $(x_1, y_1, \dots, x_n, y_n, z)$ be global coordinates in \mathbb{R}^{2n+1} . Consider the sphere

$$\mathbb{S}^{2n} = \left\{ \sum_{i=1}^n (x_i^2 + y_i^2) + z^2 = 1 \right\}$$

with equator

$$\mathbb{S}^{2n-1} = \left\{ \sum_{i=1}^n (x_i^2 + y_i^2) = 1, z = 0 \right\}.$$



We define a one form $\sigma = \frac{1}{2} \sum_{i=1}^n (x_i dy_i - y_i dx_i)$. Consider the scattering form

$$\beta = -2 \frac{dz}{z^3} \wedge \sigma + \frac{1}{z^2} d\sigma$$

restricted to \mathbb{S}^{2n} .

Proposition 3.6. $(\mathbb{S}^{2n}, \mathbb{S}^{2n-1}, \beta)$ is a scattering-symplectic manifold.

Proof. First notice that $\beta = d\left(\frac{\sigma}{z^2}\right)$. Thus β is closed. We point out that this does not make β exact as a scattering form. We are left to show that β is non-degenerate on the $2n$ -sphere. In the set $U_{x_1} := \mathbb{S}^{2n} \setminus \{x_1 = 0\}$, we have smooth coordinates $(y_1, x_2, y_2, \dots, x_n, y_n, z)$. By rewriting

$$x_1 = \sqrt{1 - y_1^2 - \sum_{i=2}^n (x_i^2 + y_i^2) - z^2}$$

and

$$dx_1 = \frac{-\left(y_1 dy_1 + \sum_{i=2}^n (x_i dx_i + y_i dy_i) + z dz\right)}{x_1},$$

we see that in U_{x_1} ,

$$\begin{aligned} \beta = & -\frac{dz}{z^3} \wedge \left(x_1 dy_1 + \frac{y_1^2 dy_1}{x_1} + \frac{y_1}{x_1} \left(\sum_{i=2}^n (x_i dx_i + y_i dy_i) \right) + \sum_{i=2}^n (x_i dy_i - y_i dx_i) \right) \\ & + \frac{1}{z^2} \frac{-\left(\sum_{i=2}^n (x_i dx_i + y_i dy_i) + z dz \right)}{x_1} \wedge dy_1 + \frac{1}{z^2} \sum_{i=2}^n (dx_i \wedge dy_i). \end{aligned}$$

The coefficient is 1 for terms of the form $\frac{1}{z^2} dx_i \wedge dy_i$ for $i = 2, \dots, n$. Thus to show non-degeneracy, it suffices to show that the coefficient of the term $\frac{dz}{z^3} \wedge dy_1$ is always nonzero. This coefficient is $-(x_1 + \frac{y_1^2}{x_1} + \frac{z^2}{x_1})$. Since, $x_1 \neq 0$, this function is always nonzero in U_{x_1} . By symmetry, this argument shows that β is non-degenerate in the sets $U_{x_i} = \mathbb{S}^{2n} \setminus \{x_i = 0\}$, and $U_{y_i} = \mathbb{S}^{2n} \setminus \{y_i = 0\}$ for $i = 1, \dots, n$. We are left to consider β at the poles where z is ± 1 , that is at the points $(0, 0, 0, \dots, 0, 0, \pm 1)$. Here, $\beta = \sum_{i=1}^n dx_i \wedge dy_i$. Thus β is non-degenerate on the $2n$ -sphere. \square

Remark 3.7. For cohomological reasons, there are no symplectic spheres in dimensions greater than 2. Similarly, Mărcuț and Orson-Torres [17] proved that a compact b-symplectic manifold (M, Z) of dimension $2n$ has a class c in $H^2(M)$ such that c^{n-1} is nonzero in $H^{2n-2}(M)$. Thus there are also no b-symplectic spheres in dimensions greater than 2.

Remark 3.8. This shows for a scattering symplectic structure to exist on a compact manifold (M, Z) , Z must admit a co-orientable contact structure. Strikingly, it also shows that sometimes this is all you need!

4. Contact Hypersurfaces

We have seen that every scattering-symplectic manifold has a co-oriented contact hypersurface. Now we will explore the opposite question: given a contact hypersurface, does it appear as the singular hypersurface of a scattering symplectic manifold? Given a co-oriented contact manifold (Z, α) , there always exists a non-compact scattering symplectic manifold (M, ω) with Z as singular hypersurface such that ω induces α on Z .

Proposition 4.1. *Let Z be a $2n-1$ dimensional contact manifold with globally defined contact form α . Let $\tilde{Z} = Z \times \mathbb{R}$ and let $\pi : \tilde{Z} \rightarrow Z$ be the obvious projection. Let x be a coordinate on \mathbb{R} . Then*

$$\omega = d\left(\frac{\pi^*\alpha}{x^2}\right)$$

*is a scattering symplectic form on (\tilde{Z}, Z) . We call (\tilde{Z}, Z, ω) the **scattering symplectization** of (Z, α) .*

Proof. It is clear by construction that ω is closed. This does not make ω exact as a scattering form. For ease of notation we will write α rather than $\pi^*\alpha$. Then

$$\omega = -\frac{2}{x^3}dx \wedge \alpha + \frac{d_Z\alpha}{x^2}.$$

To check non-degeneracy of ω , notice that

$$\omega^n = -\frac{2}{x^{2n+1}}dx \wedge \alpha \wedge (d_Z\alpha)^{n-1}.$$

A consequence of Z being contact is that $\alpha \wedge (d_Z\alpha)^{n-1} \neq 0$. Thus ω is a closed non-degenerate scattering form on (\tilde{Z}, Z) . \square

In this construction, away from $\{x = 0\}$, the one form $\frac{\alpha}{x^2}$ is a smooth primitive for ω . In particular, the vectorfield

$$V = -\frac{x}{2} \frac{\partial}{\partial x}$$

is non-zero when $x \neq 0$, is transverse to each level set $Z = \{x = c\}$ for nonzero constants $c \in \mathbb{R}$, and satisfies

$$i_V\omega = \alpha.$$

This precisely means that $\tilde{Z} \setminus \{|x| < \varepsilon\}$, for any $\varepsilon > 0$, is a disjoint union of strong symplectic fillings of the contact manifold (Z, α) .

This additional structure - a Liouville vector field V giving this relation between ω and α - is not a feature of all scattering-symplectic manifolds. Indeed, we will now show that whether or not a scattering-symplectic manifold is a strong symplectic filling in this sense can be read off of the sc-de Rham cohomology class of the scattering symplectic form.

Recall from Section 3, that we can associate to $[\omega] \in {}^{sc}H^2(M)$ a decomposition (a, b_1, b_2) where $a \in \Omega^2(Z; |N^*Z|^{-2})$, $b_1 \in H^1(Z)$, and $b_2 \in H^2(M)$. There is a tubular neighborhood of Z with a given Z defining function x giving us a trivialization $t_x : N^*Z \rightarrow \mathbb{R}$. By Proposition 3.3, this defines a smooth contact form $\alpha = (t_x)_*(a) \in \Omega^1(Z)$, $\beta_1 \in b_1$, and $\beta_2 \in b_2$ such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2.$$

In the following propositions we will always be working with ω in such a tubular neighborhood.

Proposition 4.2. *Let (M, Z, ω) be a scattering-symplectic manifold with singular contact hypersurface (Z, α) . If $[\omega]$ has cohomology decomposition (a, b_1, b_2) with b_1 or $b_2 \neq 0$, then for $\varepsilon > 0$ small, $(M \setminus \{|x| < \varepsilon\}, \omega)$ is not a strong symplectic filling of (Z, α) .*

The following lemma gives us a normal form for strong convex and strong concave fillings in a neighborhood of the boundary.

Lemma 4.3. *If (M, ω) is a strong convex symplectic filling of (Z, ξ) , then for some $c > 0$, there exists a collar neighborhood $Z \times [0, c)_r$ of Z on which*

$$\omega = d(e^{-r}\alpha)$$

and, given the projection $p : Z \times [0, c) \rightarrow Z$, $\alpha = p^(\tilde{\alpha})$ for an $\tilde{\alpha}$ satisfying $\ker \tilde{\alpha} = \xi$.*

If (M, ω) is a strong concave symplectic filling of (Z, ξ) , then for some $c > 0$, there exists a collar neighborhood $Z \times [0, c)_r$ of Z on which

$$\omega = d(e^r\alpha)$$

and, given the projection $p : Z \times [0, c) \rightarrow Z$, $\alpha = p^(\tilde{\alpha})$ for an $\tilde{\alpha}$ satisfying $\ker \tilde{\alpha} = \xi$.*

The proof of this lemma can be found in the Appendix. We are now prepared to prove Proposition 4.2.

Proof. By Proposition 3.3, there is a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of Z such that

$$\omega = \frac{dx}{x^3} \wedge (\alpha + x^2\beta_1) - \frac{d\alpha}{2x^2} + \beta_2$$

for $\alpha, \beta_1, \beta_2 \in \Omega^*(Z)$. Assume, for a contradiction, that $M \setminus \{x < \varepsilon\}$ for $\varepsilon > 0$ is a strong symplectic filling of (Z, α) .

By Lemma 4.3, there is a smooth function $f \in C^\infty(\tau)$ such that

$$\omega = d(f\alpha) = \partial_x f dx \wedge \alpha + d_Z f \wedge \alpha + f d\alpha.$$

Thus

$$\frac{dx}{x^3} \wedge (\alpha + \beta_1 x^2) = \partial_x f dx \wedge \alpha \text{ and } \frac{\alpha}{x^3} + \frac{\beta_1}{x} = \partial_x f \alpha.$$

We can solve for β_1 ,

$$\beta_1 = (x\partial_x f - \frac{1}{x^2})\alpha.$$

Since β_1 is closed,

$$0 = d\beta_1 = (x\partial_x f - \frac{1}{x^2})d\alpha + (\partial_x f + x\partial_{xx}f + \frac{2}{x^3})dx \wedge \alpha + x d_Z(\partial_x f) \wedge \alpha.$$

Thus

$$\partial_x f + x \partial_{xx} f = -\frac{2}{x^3}$$

with solution $f = -\frac{1}{2x^2}$.

Then

$$\omega = d\left(-\frac{1}{2x^2}\alpha\right) = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2} \text{ and } \beta_1 = \beta_2 = 0.$$

Thus we have reached a contradiction. \square

On the other hand, if $b_1, b_2 = 0$ in a cohomological decomposition of a scattering symplectic form $[\omega]$, then we always have the additional structure of a strong symplectic filling. And in fact, that filling is always convex, meaning the Liouville vector field points outward at Z .

Proposition 4.4. *Let (M, Z, ω) be a scattering-symplectic manifold with singular contact hypersurface (Z, α) . If $[\omega]$ has cohomology decomposition $(a, 0, 0)$, then for $\varepsilon > 0$ small, $(M \setminus \{|x| < \varepsilon\}, \omega) = (M_{x \geq \varepsilon}, \omega) \cup (M_{x \leq -\varepsilon}, \omega)$ is a collection of symplectic manifolds each with contact boundary (Z, α) such that ω is a convex strong symplectic filling of α .*

Proof. Choose a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of Z as in Proposition 3.3. Then

$$\omega|_\tau = \frac{dx}{x^3} \wedge \alpha - \frac{d\alpha}{2x^2}.$$

Define $M_{x \geq \varepsilon}$ to be the connected component of $M \setminus (Z \times (-\varepsilon, \varepsilon))$ containing $Z \times \{\varepsilon\}$ and let its symplectic form be the scattering symplectic form on M restricted to $M_{x \geq \varepsilon}$. Similarly, define $M_{x \leq -\varepsilon}$ to be the connected component of $M \setminus (Z \times (-\varepsilon, \varepsilon))$ containing $Z \times \{-\varepsilon\}$ and let its symplectic form be the scattering symplectic form on M restricted to $M_{x \leq -\varepsilon}$.

Let $V = -\frac{x}{2}\partial_x$. Notice for all points in $Z \times \{\varepsilon\}$ and $Z \times \{-\varepsilon\}$ that V is tranverse to Z . Next, observe that $i_V \omega = \frac{\alpha}{x^2}$. Thus $\mathcal{L}_V \omega = di_V \omega = \omega$. Notice that ω on $M_{x \geq \varepsilon}$ and ω on $M_{x \leq -\varepsilon}$ induce the same contact form. In particular,

$$\frac{\alpha}{x^2} \Big|_{Z \times \{\varepsilon\}} = \frac{\alpha}{(\varepsilon)^2} = \frac{\alpha}{(-\varepsilon)^2} = \frac{\alpha}{x^2} \Big|_{Z \times \{-\varepsilon\}}.$$

Further, notice that $V|_{Z \times \{\varepsilon\}} = -\frac{\varepsilon}{2}\partial_x$ is an outward pointing vector. Similarly, $V|_{Z \times \{-\varepsilon\}} = -\frac{-\varepsilon}{2}\partial_x$ is outward pointing. Thus both fillings are convex. \square

4.1. Symplectic Gluing. We will demonstrate how to glue strong symplectic fillings along a common boundary. In particular, by allowing scattering-symplectic structures, we can glue convex fillings to convex fillings and by allowing folded-symplectic structures, we can glue concave fillings to concave fillings.

We will begin by recalling how a strong concave and strong convex symplectic filling are glued to form a symplectic manifold; see for example Theorem 5.4, [7].

Proposition 4.5. *Let (M_1, ω_1) and (M_2, ω_2) be a strong convex and strong concave symplectic filling respectively of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$.*

Proof. By Lemma 4.3, since ω_1 is a strong convex symplectic filling of (Z, ξ) , there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which $\omega_1 = d(e^{-r_1}\alpha)$ for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, because (M_2, ω_2) is a strong concave symplectic filling, we have $Z \times [0, c_2)$ a tubular neighborhood of Z in M_2 on which $\omega_2 = d(e^{r_2}\alpha)$ for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1 = c_2 = c$.

We attach a collar neighborhood $Z \times (-c, 0)_{r_1}$ to M_1 and a collar neighborhood $Z \times (-c, 0)_{r_2}$ to M_2 . The union $M_1 \cup_Z M_2$ is formed from identifying $Z \times (-c, c)_{r_1}$ with $Z \times (-c, c)_{r_2}$ by mapping Z to itself and setting $r_1 = -r_2 = r$. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on M_1 and M_2 respectively.

We define a symplectic form on $M_1 \cup_Z M_2$ by $\omega = d(e^r\alpha)$. We interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, c)_{r_1})$. Similarly, we interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, c)_{r_2})$. \square

Next, we introduce a method for gluing two strong convex symplectic fillings by using a scattering-symplectic structure.

Theorem 4.6. *Let (M_1, ω_1) and (M_2, ω_2) be strong convex symplectic fillings of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a scattering symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$, and the singular hypersurface of ω is Z .*

Proof. By Lemma 4.3, since ω_1 is a strong convex symplectic filling of (Z, ξ) , there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which

$$\omega_1 = d(e^{-r_1}\alpha)$$

for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, let $Z \times [0, c_2)$ be a tubular neighborhood of Z in M_2 on which

$$\omega_2 = d(e^{-r_2}\alpha)$$

for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1, c_2 > 2$.

The union of M_1 to M_2 at Z is formed from the disjoint union

$$(M_1 \setminus \{Z \times [0, 1/2]_{r_1}\}) \sqcup (M_2 \setminus \{Z \times [0, 1/2]_{r_2}\})$$

by identifying $Z \times (1/2, 2)_{r_1}$ with $Z \times (1/2, 2)_{r_2}$ by mapping Z to itself and setting $r_1 = \frac{1}{r_2}$. In other words, we have identified the annulus $Z \times (1/2, 2)_{r_1}$ with the annulus $Z \times (1/2, 2)_{r_2}$ by inverting the first about $r_1 = 1$ and gluing it to the latter. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on $M_1 \setminus \{Z \times [0, 1/2]_{r_1}\}$ and $M_2 \setminus \{Z \times [0, 1/2]_{r_2}\}$ respectively.

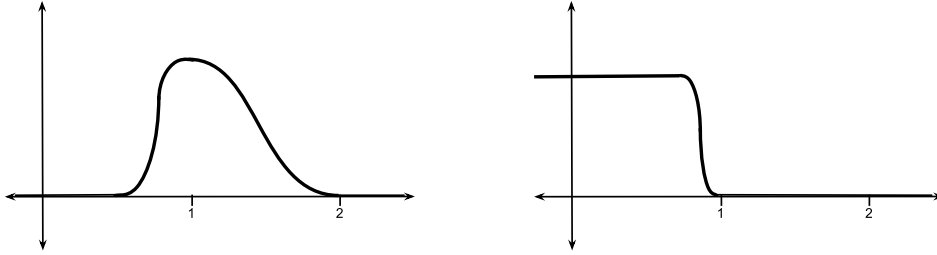
Next, we will define a scattering-symplectic form ω on $M_1 \cup_Z M_2$ with singular locus $Z \times \{r_1 = r_2 = 1\}$. Let $\gamma = e^{-r_1}\alpha$ denote a primitive for ω_1 and let $\tilde{\gamma} = e^{-r_2}\alpha$ denote a primitive for ω_2 . We define

$$\omega = d\left(\left(\frac{\phi(r_1)}{(r_1 - 1)^2} + \psi(r_1)\right)\tilde{\gamma}\right) + d\left(\left(\frac{\phi(r_2)}{(r_2 - 1)^2} + \psi(r_2)\right)\gamma\right) \quad (4.1)$$

where

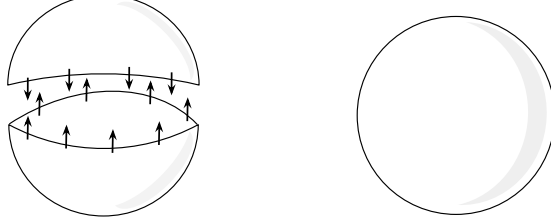
- $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth bump function supported in $(1/2, 2)$ and
- $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $(-\infty, 1)$ such that $\psi|_{(-\infty, 7/8)} \equiv 1$.

Figure 4.7. The functions ϕ and ψ from (6.1) and (6.2).



By definition, ω is closed. Since $\psi(r_1) = 1$ for $r_1 \leq 7/8$, we interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, 2)_{r_2})$. Similarly, since $\psi(r_2) = 1$ for $r_2 \leq 7/8$, we interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, 2)_{r_1})$. By Lemma 6.2, whose statement and proof can be found in Section 6, functions ϕ and ψ exist that make ω into a non-degenerate scattering form. \square

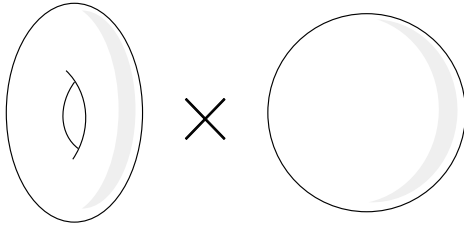
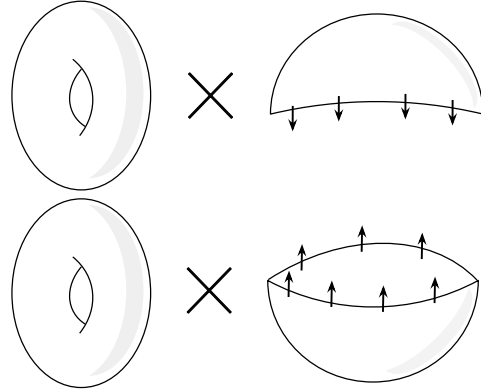
Remark 4.8. Because \mathbb{D}^{2n} with the standard symplectic form ω_{st} is a strong convex symplectic filling of the unit sphere \mathbb{S}^{2n-1} with standard contact structure ξ_{st} , Theorem 4.6 provides an alternate way of constructing the scattering symplectic spheres described in Proposition 3.6.



Theorem 4.6 provides us with a treasure trove of additional examples, particularly in dimension 4 where constructing strong convex symplectic fillings has been an industry in its own right. For a certainly incomplete list, see [6, 9, 10], or [20]. For the sake of brevity, we will limit our attention to a couple of examples.

Example 4.9. The pair $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$ is scattering symplectic.

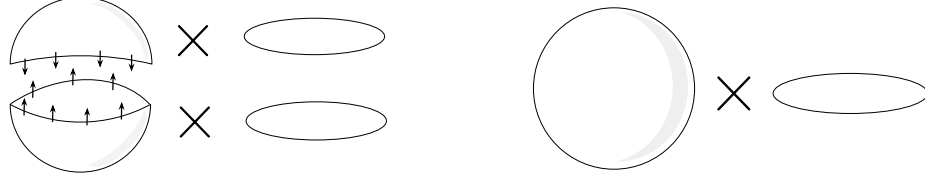
Let (q_1, q_2) be coordinates for the torus \mathbb{T}^2 and let (p_1, p_2) be coordinates for the disk \mathbb{D}^2 . Then $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$ is a symplectic form on $\mathbb{T}^2 \times \mathbb{D}^2$. We can rewrite this expression using polar coordinates by setting $p_1 = r \cos \theta$ and $p_2 = r \sin \theta$. Then $\gamma = r \cos \theta \, dq_1 + r \sin \theta \, dq_2$ is a primitive for ω near the boundary $\partial(\mathbb{T}^2 \times \mathbb{D}^2)$. At the boundary, $\gamma|_{r=1} = \cos \theta \, dq_1 + \sin \theta \, dq_2$ is a contact form on $\mathbb{T}^2 \times \mathbb{S}^1 = \mathbb{T}^3$. Notice that $i_{r\partial_r} d\gamma = \gamma$ for outward pointing normal vector $r\partial_r$.



Thus $(\mathbb{T}^2 \times \mathbb{D}^2, \omega)$ is a strong convex symplectic filling of the torus \mathbb{T}^3 with contact structure $\gamma|_{r=1} = 0$. As described in Theorem 4.6, we construct the union of $\mathbb{T}^2 \times \mathbb{D}^2$ with itself at $\partial(\mathbb{T}^2 \times \mathbb{D}^2)$. Thus the pair $(\mathbb{T}^2 \times \mathbb{S}^2, \mathbb{T}^3)$, where \mathbb{T}^3 is identified as $\mathbb{T}^2 \times \mathbb{S}^1$ and the factor \mathbb{S}^1 is the equator of \mathbb{S}^2 , admits a scattering symplectic structure.

Example 4.10. The pair $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$ is scattering symplectic.

Let (x, y, z) be the standard Euclidean coordinates for \mathbb{D}^3 and let θ be a coordinate for \mathbb{S}^1 . The manifold $\mathbb{D}^3 \times \mathbb{S}^1$ admits the symplectic form $\omega = 2dx \wedge dy + dz \wedge d\theta$. Then $\gamma = xdy - ydx + zd\theta$ is a primitive for ω . The radial vector field in \mathbb{D}^3 , $R = x\partial_x + y\partial_y + z\partial_z$, is an outward pointing normal vector at the boundary $\partial(\mathbb{D}^3 \times \mathbb{S}^1)$. Further $i_R d\gamma = \gamma$ and $\gamma|_{\partial(\mathbb{D}^3 \times \mathbb{S}^1)} = 0$ defines a contact structure on $\mathbb{S}^2 \times \mathbb{S}^1$. Thus $(\mathbb{D}^3 \times \mathbb{S}^1, \omega)$ is a strong convex symplectic filling of $\mathbb{S}^2 \times \mathbb{S}^1$ with contact structure $\gamma = 0$.



By Theorem 4.6, we construct the union of $\mathbb{D}^3 \times \mathbb{S}^1$ with itself at $\partial(\mathbb{D}^3 \times \mathbb{S}^1)$. Then the pair $(\mathbb{S}^3 \times \mathbb{S}^1, \mathbb{S}^2 \times \mathbb{S}^1)$, where the factor \mathbb{S}^2 in $\mathbb{S}^2 \times \mathbb{S}^1$ is identified as the equator of \mathbb{S}^3 , admits a scattering symplectic structure.

By expanding the symplectic category to allow scattering symplectic structures, we can overcome the obstacle preventing convex fillings from being glued to other convex fillings. In the next theorem, we show that folded symplectic structures can similarly overcome this obstacle for concave fillings, allowing a concave filling to be glued to another concave filling.

Recall that a folded symplectic manifold (M^{2n}, Z, ω) is a $2n$ -dimensional manifold M equipped with a closed two-form ω that is non-degenerate except on a hypersurface Z , called the folding hypersurface, where there exist coordinates such that locally $Z = \{x_1 = 0\}$ and

$$\omega = x_1 dx_1 \wedge dy_1 + \sum_{i=2}^n dx_i \wedge dy_i.$$

In section 6 of [3], Ana Cannas da Silva, Victor Guillemin, and Christopher Woodward prove that given any two compact oriented 2d-dimensional symplectic manifolds W_1, W_2 with common boundary Z , their union over their boundary $W_1 \cup_Z W_2$ admits a folded-symplectic structure. We will consider the special case when the two manifolds W_1 and W_2 are strong concave symplectic fillings of a contact boundary (Z, α) . We will prove that $W_1 \cup_Z W_2$ can be endowed with a folded-symplectic structure that preserves this strong concavity on either side of the hypersurface Z .

Theorem 4.11. *Let (M_1, ω_1) and (M_2, ω_2) be strong concave symplectic fillings of (Z, ξ) . Then $M_1 \cup_Z M_2$, the union of M_1 to M_2 at Z , has a folded symplectic structure ω such that $\omega|_{M_1 \setminus Z} \simeq \omega_1$, $\omega|_{M_2 \setminus Z} \simeq \omega_2$, and the folding hypersurface of ω is Z .*

Proof. Since ω_1 is a strong concave symplectic filling, by Lemma 4.3, there exists $Z \times [0, c_1)$ a tubular neighborhood of Z in M_1 on which $\omega_1 = d(e^{r_1}\alpha)$ for r_1 the coordinate for the interval $[0, c_1)$ and where $\ker \alpha = \xi$. Similarly, let $Z \times [0, c_2)$ be a tubular neighborhood of Z in M_2 on which $\omega_2 = d(e^{r_2}\alpha)$ for r_2 the coordinate of the interval $[0, c_2)$. Without loss of generality, assume $c_1, c_2 > 2$.

We attach a collar neighborhood $Z \times (-2, 0)_{r_1}$ to M_1 and a collar neighborhood $Z \times (-2, 0)_{r_2}$ to M_2 .

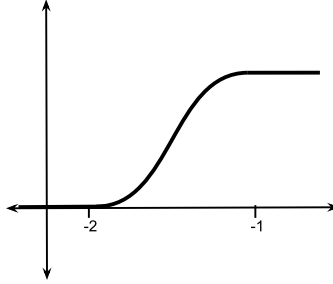
The union $M_1 \cup_Z M_2$ is formed from identifying $Z \times (-2, 2)_{r_1}$ with $Z \times (-2, 2)_{r_2}$ by mapping Z to itself and setting $r_1 = -r_2$. The smooth structure on $M_1 \cup_Z M_2$ is obtained from the charts on M_1 and M_2 respectively.

Next, we will define a folded symplectic form ω on $M_1 \cup_Z M_2$ with folding hypersurface $Z \times \{r_1 = r_2 = 0\}$. We define

$$\omega = d(\psi(r_1)e^{r_1}\alpha) + d(\psi(r_2)e^{r_2}\alpha) \quad (4.2)$$

where $\psi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function supported in $(-2, \infty)$ with $\psi|_{(-1, \infty)} \equiv 1$.

Figure 4.12. The function ψ from (6.3).



By definition, ω is closed. Since $\psi(r_1) = 1$ for $r_1 > -1$, we interpret ω to extend as ω_1 into $M_1 \setminus (Z \times [0, 2)_{r_1})$. Similarly, since $\psi(r_2) = 1$ for $r_2 > -1$, we interpret ω to extend as ω_2 into $M_2 \setminus (Z \times [0, 2)_{r_2})$. By Lemma 6.3, which can be found in Section 6, such a function ψ exists and ω is a folded symplectic form. \square

Because $\mathbb{R}^{2n} \setminus \mathbb{D}^{2n}$ with the standard symplectic form is a concave symplectic filling of \mathbb{S}^{2n-1} with the standard contact form, this construction immediately gives us a folded-symplectic connect sum over the sphere with its standard contact structure. This construction is due to Cannas da Silva, Guillemin, and Woodward; see Example (2), Section 3 in [3].

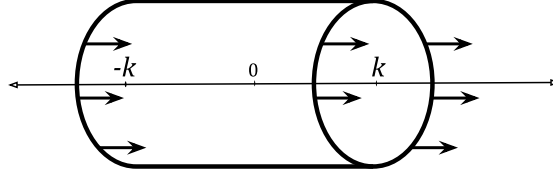
Corollary 4.13 (Cannas da Silva-Guillemin-Woodward). *Given $2n$ dimensional symplectic manifolds (M_1, ω_1) and (M_2, ω_2) , their connect sum $M_1 \# M_2$ is folded-symplectic with folding hypersurface \mathbb{S}^{2n-1} with its standard contact structure.*

By combining Theorems 4.6 and 4.11, we can construct examples of scattering-folded symplectic manifolds:

Definition 4.14. A *scattering-folded symplectic manifold* (M, Z_{sc}, Z_f, ω) is a manifold M , with two distinct hypersurfaces Z_{sc} and Z_f , equipped with a two form ω that is symplectic everywhere except for on a singular hypersurface Z_{sc} where it is a scattering symplectic form and on a folding hypersurface Z_f where it is a folded symplectic form.

Example 4.15. Recall [15] that a *symplectic cone* is a triple (M, ω, X) of a manifold M with a symplectic form ω and a vector field X such that X generates a proper action of the real numbers on M and $\mathcal{L}_X \omega = \omega$. Any symplectic cone is of the form $B \times \mathbb{R}$ where B is a co-oriented contact manifold with contact form α . In fact, all symplectic cones can be written as $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ where t is the \mathbb{R} coordinate.

Let $(B \times \mathbb{R}, d(e^t \alpha), \frac{\partial}{\partial t})$ be any symplectic cone of a co-oriented contact manifold (B, α) . We will truncate $B \times \mathbb{R}$ by considering $B \times [-k, k]$ for any real number $k > 0$.



At the level set $t = k$, the vector field $\frac{\partial}{\partial t}$ is outward pointing and satisfies $i_{\frac{\partial}{\partial t}} \omega = \alpha$. Thus at $t = k$, we have a strong convex symplectic filling of (B, α) . At the level set $t = -k$, the vector field $\frac{\partial}{\partial t}$ is inward pointing and satisfies $i_{\frac{\partial}{\partial t}} \omega = \alpha$. Thus at $t = -k$, we have a strong concave symplectic filling of (B, α) .

By Theorems 4.6 and 4.11, we can glue $B \times [-k, k]$ to a copy of itself to form the scattering-folded symplectic manifold $(B \times \mathbb{S}^1, B \times \{k\}, B \times \{-k\}, \omega)$.

Our next example provides an explicit description of a scattering-folded symplectic manifold.

Example 4.16. The triple $(\mathbb{T}^{2n}, \cup_{2m} \mathbb{T}^{2n-1}, \cup_{2m} \mathbb{T}^{2n-1})$ is scattering-folded symplectic for all $n, m \in \mathbb{N}$.

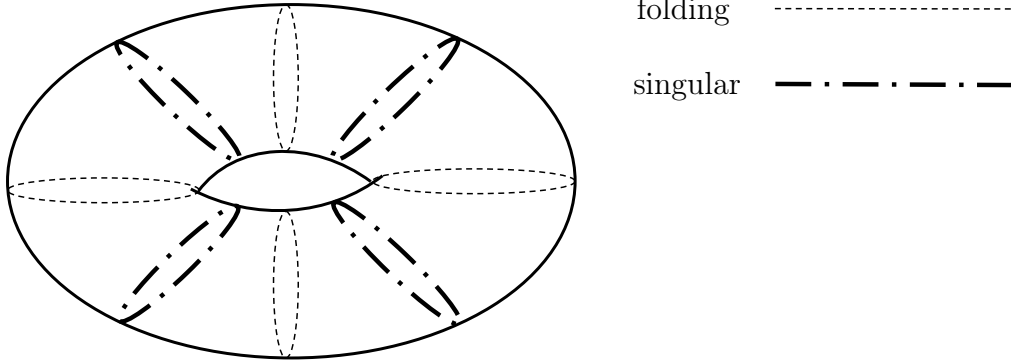
In [1], Frédéric Bourgeois defines a contact form β on all odd dimensional tori \mathbb{T}^{2n-1} . We identify \mathbb{T}^{2n} with $\mathbb{T}^{2n-1} \times \mathbb{S}^1$ and denote the angular coordinate on \mathbb{S}^1 by θ . We define a scattering-folded form on \mathbb{T}^{2n} by

$$\omega_m = d \left(\frac{\beta}{\sin^2(m\theta)} \right) = \frac{-2m \cos(m\theta) d\theta \wedge \beta}{\sin^3(m\theta)} + \frac{d\beta}{\sin^2(m\theta)}$$

for any $m \in \mathbb{N}$.

Then for each zero z of $\sin(m\theta)$ in $[0, 2\pi)$, we have a singular hypersurface $\mathbb{T}^{2n-1} \times \{z\}$. Since there are $2m$ zeroes of $\sin(m\theta)$ in $[0, 2\pi)$, $Z_{sc} = \cup_{2m} \mathbb{T}^{2n-1}$. Similarly, for each zero z of $\cos(m\theta)$ in $[0, 2\pi)$, we have a folding hypersurface $\mathbb{T}^{2n-1} \times \{z\}$. Since there are $2m$ zeroes of $\cos(m\theta)$ in $[0, 2\pi)$, $Z_f = \cup_{2m} \mathbb{T}^{2n-1}$.

Figure 4.17. $(\mathbb{T}^2, \cup_4 \mathbb{S}^1, \cup_4 \mathbb{S}^1)$.



We equip \mathbb{T}^2 with the form

$$d\left(\frac{d\theta_1}{\sin^2(2\theta_2)}\right) = \frac{-4 \cos(2\theta_2) d\theta_2 \wedge d\theta_1}{\sin^3(2\theta_1)}.$$

The folding hypersurfaces occur at $\theta_2 = \pi/4, 3\pi/4, 5\pi/4$, and $7\pi/4$. The singular hypersurfaces occur at $\theta_2 = 0, \pi/2, \pi$, and $3\pi/2$.

5. Scattering Poisson Geometry

A scattering-Poisson structure is dual to a scattering-symplectic structure. In this section we will explore these structures utilizing the language of Poisson geometry.

Definition 5.1. A Poisson manifold (M, π) is *scattering-Poisson* if there exists an oriented hypersurface $Z \subset M$ such that there is a bi-vector $\pi_{sc} \in \bigwedge^2({}^{sc}TM)$ with $\rho(\pi_{sc}) = \pi$.

It is assumed that π_{sc} is non-degenerate unless otherwise stated.

The following lemma gives an alternative definition of a scattering Poisson manifold.

Lemma 5.2. *Given a Poisson manifold (M^{2n}, π) , if $(\wedge^n \pi)^{-1}(0)$ is an oriented hypersurface Z , then there exists a bi-vector $\pi_b \in \bigwedge^2({}^bTM)$ such that $\rho(\pi_b) = \pi$. Consider any Taylor series expansion of the associated bi-vector π_b at Z . If the first non-zero coefficient $\tilde{\pi}_b$ is in degree 2, and $\tilde{\pi}_b$ is full rank for all $p \in Z$, then (M^{2n}, π) is Scattering-Poisson.*

Proof. Let (M^{2n}, π) be a Poisson manifold as specified. Let x be a $Z = (\wedge^n \pi)^{-1}(0)$ defining function. It suffices to check the statement locally, since we can use a partition of unity to define a global bi-vector π_{sc} such that $\rho(\pi_{sc}) = \pi$. Since there is $\pi_b \in \wedge^2(bTM)$ such that $\rho(\pi_b) = \pi$, then there exist local coordinates such that $\pi = x \sum \frac{\partial}{\partial x} \wedge A + B$ where A and B are a locally defined vector field and bi-vector respectively on Z . Since any Taylor series expansion of π_b at Z satisfies that the first non-zero coefficient $\tilde{\pi}_b$ is in degree 2, and $\tilde{\pi}_b$ is full rank for all $p \in Z$, then $A = x^2 \tilde{A}$ for a non-zero vector field \tilde{A} on Z and $B = x^2 \tilde{B}$ for a non-zero bi-vector \tilde{B} on Z . Thus

$$\pi = x^3 \frac{\partial}{\partial x} \wedge \tilde{A} + x^2 \tilde{B}$$

can be realized as a section π_{sc} of $\wedge^2(scTM)$. Because $\tilde{\pi}_b$ is full rank for all $p \in Z$, π_{sc} is non-degenerate. \square

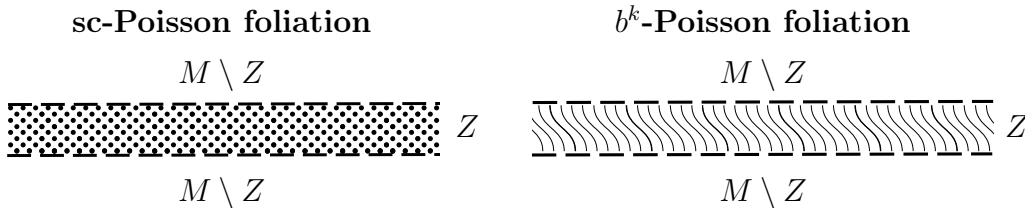
By dualizing the scattering-symplectic form as in Proposition 3.2, for any non-degenerate scattering-Poisson manifold (M, π) , for all $p \in Z$ there exists a coordinate chart $(U, x_1, y_1, \dots, x_n, y_n)$ centered at p such that on U , the hypersurface Z is locally defined by $\{x_1 = 0\}$, and

$$\pi = x_1^3 \frac{\partial}{\partial y_1} \wedge \frac{\partial}{\partial x_1} + x_1^2 \frac{\partial}{\partial y_1} \wedge \left(\sum_{i=2}^n y_i \frac{\partial}{\partial y_i} + x_i \frac{\partial}{\partial x_i} \right) + x_1^2 \sum_{i=2}^n \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

5.0.1. Symplectic foliations. Every Poisson structure on a manifold induces a foliation by symplectic manifolds. The symplectic foliation for a non-degenerate scattering Poisson structure contains the open leaves $M \setminus Z$, locally given by $\{x < 0\}$ and $\{x > 0\}$. The individual points of the hyperplane Z are zero dimensional symplectic leaves.

Recall from [12] and [25], in the case of b - and b^k -Poisson structures, the symplectic foliation contains the open leaves $M \setminus Z$, and a regular codimension 1 foliation of the hypersurface Z .

Figure 5.3.



Note that the Hamiltonian vector fields associated to a scattering-Poisson manifold are all zero at Z . However, as we will show in the following discussion,

π contains information about the induced contact structure on the degeneracy hypersurface. Since contact structures are maximally non-integrable, it is fitting that the associated symplectic foliation is maximally trivial.

5.1. Poisson cohomology and \mathcal{A} -Poisson cohomology. Recall ([5], p. 39) for a general Poisson manifold (M, π) , the Poisson cohomology $H_\pi^*(M)$ is defined as the cohomology groups of the Lichnerowicz complex: This complex is formed using $\mathcal{V}^k(M) := \mathcal{C}^\infty(M; \wedge^k TM)$, smooth multivector fields on M .

$$\dots \rightarrow \mathcal{V}^{k-1}(M) \xrightarrow{d_\pi} \mathcal{V}^k(M) \xrightarrow{d_\pi} \mathcal{V}^{k+1}(M) \rightarrow \dots$$

differential

$$d_\pi : \mathcal{V}^k(M) \rightarrow \mathcal{V}^{k+1}(M)$$

is defined as

$$d_\pi = [\pi, \cdot],$$

where $[\cdot, \cdot]$ is the Schouten bracket extending the standard Lie bracket on vector fields $\mathcal{V}^1(M)$.

Because Poisson cohomology is quite challenging to compute, there are only very select cases where the answer is known. In the case of a symplectic manifold where the Poisson bi-vector is non-degenerate, the Poisson cohomology is isomorphic to the de Rham cohomology. The non-degeneracy of π allows us to define an isomorphism $T^*M \rightarrow TM$ that provides this isomorphism in cohomology: $H^p(M) \simeq H_\pi^p(M)$. In the case where π can be realized as a non-degenerate bi-vector on an algebroid, we recover the analogous isomorphism, but on different cohomologies.

Let ${}^{\mathcal{A}}TM$ be an algebroid over (M, Z) constructed by rescaling the tangent bundle TM at Z ; for instance, any of the bundles constructed in Examples 2.3 through 2.7. We can define the ${}^{\mathcal{A}}TM$ -Poisson cohomology of a non-degenerate ${}^{\mathcal{A}}TM$ -Poisson manifold (M, Z, π) . Let $\pi_{\mathcal{A}} \in \Gamma(\wedge {}^{\mathcal{A}}TM)$ be the non-degenerate section that satisfies $\rho(\pi_{\mathcal{A}}) = \pi$. We denote the smooth \mathcal{A} -multivector fields by

$${}^{\mathcal{A}}\mathcal{V}^k(M) = \mathcal{C}^\infty(M; \wedge^k ({}^{\mathcal{A}}TM)).$$

The operator

$$d_{\pi_{\mathcal{A}}} = [\pi_{\mathcal{A}}, \cdot]$$

is a differential on this subalgebra. The ${}^{\mathcal{A}}TM$ -Poisson cohomology ${}^{\mathcal{A}}H_\pi^*(M)$ is the cohomology of the complex

$$\dots \rightarrow {}^{\mathcal{A}}\mathcal{V}^{k-1}(M) \xrightarrow{d_{\pi_{\mathcal{A}}}} {}^{\mathcal{A}}\mathcal{V}^k(M) \xrightarrow{d_{\pi_{\mathcal{A}}}} {}^{\mathcal{A}}\mathcal{V}^{k+1}(M) \rightarrow \dots$$

Proposition 5.4. *Let (M, Z, ω) be an ${}^{\mathcal{A}}TM$ -symplectic manifold, and π the corresponding non-degenerate ${}^{\mathcal{A}}TM$ -Poisson structure. Then, the ${}^{\mathcal{A}}TM$ -Poisson cohomology ${}^{\mathcal{A}}H_\pi^*(M)$ is isomorphic to the ${}^{\mathcal{A}}TM$ -de Rham cohomology ${}^{\mathcal{A}}H^*(M)$.*

The details of this proof are identical to the standard symplectic case which can be found in Section 2.1.3 of [5] or to the b -symplectic case which can be found in the proof of Theorem 30 in [12].

Example 5.5. If (M, π) is a non-degenerate b -Poisson manifold, then

$${}^b H_\pi^p(M) \stackrel{[12]}{\simeq} {}^b H^p(M) \stackrel{[21]}{\simeq} H^p(M) \oplus H^{p-1}(Z).$$

If (M, π) is a non-degenerate b^k -Poisson manifold, then

$${}^{b^k} H_\pi^p(M) \simeq {}^{b^k} H^p(M) \stackrel{[25]}{\simeq} H^p(M) \oplus (H^{p-1}(Z))^k.$$

If (M, π) is a non-degenerate sc -Poisson manifold, then

$${}^{sc} H_\pi^p(M) \simeq {}^{sc} H^p(M) \stackrel{Thm\ 2.15}{\simeq} H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z; |N^*Z|^{-p}).$$

Thus far we have constructed the following diagram of complexes.

$$\begin{array}{ccc} ({}^{\mathcal{A}}\mathcal{V}^*, d_{\pi_{\mathcal{A}}}) & \xrightarrow[\simeq]{\omega^b} & ({}^{\mathcal{A}}\Omega^*, d) \\ \downarrow i & & \\ (\mathcal{V}^*, d_\pi) & & \end{array}$$

In the b -Poisson setting, Ioan Mărcuț and Boris Osorno Torres [16] showed that the inclusion map $i : ({}^b\mathcal{V}^*, d_{\pi_b}) \rightarrow (\mathcal{V}^*, d_\pi)$ induces an isomorphism in cohomology and thus $H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z)$. However, for general ${}^{\mathcal{A}}TM$ -Poisson structures this will not be the case. In order to compute the Poisson cohomology of an ${}^{\mathcal{A}}TM$ -symplectic manifold (M, Z, ω) , we construct a complex $({}^{\mathcal{R}}\Omega^*, d)$, called the Rigged de Rham complex, that is isomorphic to the Lichnerowicz complex, but that is much more tractable when computing cohomology groups.

$$\begin{array}{ccc} ({}^{\mathcal{A}}\mathcal{V}^*, d_{\pi_{\mathcal{A}}}) & \xrightarrow[\simeq]{\omega^b} & ({}^{\mathcal{A}}\Omega^*, d) \\ \downarrow i & & \downarrow i \\ (\mathcal{V}^*, d_\pi) & \xrightarrow[\simeq]{\omega^b} & ({}^{\mathcal{R}}\Omega^*, d) \end{array}$$

We define the Rigged lie algebroid \mathcal{R} associated to an ${}^{\mathcal{A}}TM$ -symplectic structure by describing the Lie co-algebroid \mathcal{R}^* and the \mathcal{R} -de Rham complex.

Definition 5.6. Given an ${}^{\mathcal{A}}TM$ -symplectic manifold (M, Z, ω) , the dual \mathcal{A} -rigged bundle \mathcal{R}^* is the image $\omega^b(TM)$. The \mathcal{A} -rigged de-Rham forms are ${}^{\mathcal{R}}\Omega^p(M) = \Gamma(\wedge^p \mathcal{R}^*)$, smooth sections of the p -th exterior power of \mathcal{R}^* . This complex has exterior derivative d given by extending the standard smooth differential on $M \setminus Z$ to M .

5.2. Poisson cohomology of a scattering-Poisson manifold. Consider a non-degenerate scattering Poisson manifold (M, Z, π) and let ω be the corresponding scattering-symplectic form.

We will now give an explicit description of the scattering-rigged algebroid. Given a choice of Z defining function x , ω induces a contact form α on Z . Let R be the Reeb vector field associated to α . That is the non-vanishing vector field R on Z such that

$$\alpha(R) = 1 \text{ and } i_R d\alpha = 0.$$

Then TZ splits as $\mathbb{R}\langle R \rangle \oplus \mathbb{R}\langle \ker \alpha \rangle$ by

$$V \mapsto \alpha(V)R + (V - \alpha(V)R).$$

Consider the Lie algebroid \mathcal{A} whose space of sections D is

$$\{u \in \mathcal{C}^\infty(M; TM) : u|_Z \in \mathcal{C}^\infty(M; \ker \alpha)\}.$$

By Theorem 2.2 this is in fact a Lie algebroid. We rescale \mathcal{A} by taking the \mathcal{A} -vector fields that are zero at Z . Let \mathcal{B} be the algebroid whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; \mathcal{A}) : u|_Z = 0\}.$$

We complete one final rescaling to arrive at the rigged algebroid. The scattering rigged algebroid \mathcal{R} is the algebroid whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; \mathcal{B}) : u|_Z = 0\}. \quad (5.1)$$

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$, in Z such that $\alpha = dr + \sum_{i=1}^m s_i dt_i$,

the local sections of \mathcal{R} are smooth linear combinations of

$$x^3 \frac{\partial}{\partial x}, x^3 \frac{\partial}{\partial r}, x^2 \frac{\partial}{\partial s_1}, \dots, x^2 \frac{\partial}{\partial s_m}, x^2 \left(s_1 \frac{\partial}{\partial r} - \frac{\partial}{\partial t_1} \right), \dots, x^2 \left(s_m \frac{\partial}{\partial r} - \frac{\partial}{\partial t_m} \right).$$

Notice that we can locally identify \mathcal{R}^* as the span of

$$\frac{dx}{x^3}, \frac{\alpha}{x^3}, \frac{ds_1}{x^2}, \dots, \frac{ds_m}{x^2}, \frac{dt_1}{x^2}, \dots, \frac{dt_m}{x^2}.$$

Lemma 5.7. *The Poisson cohomology of a non-degenerate scattering-Poisson manifold (M, Z, π) is isomorphic to the de Rham cohomology ${}^{\mathcal{R}}H^*(M)$ of the scattering rigged algebroid \mathcal{R} identified as (5.1).*

Proof. Let ω be the scattering symplectic form dual to π . We define a map $\bar{\omega} : TM \rightarrow \mathcal{R}^*$ by $v \mapsto i_v \omega$. For all $p \in M \setminus Z$, $\mathcal{R}_p^* \simeq T_p^*M$ and $\bar{\omega}_p$ is a symplectic form. Thus for all $p \in M \setminus Z$, we have that $\bar{\omega}_p$ is an isomorphism. For $p \in Z$, it follows from ω being non-degenerate as a scattering 2-form that $\bar{\omega}$ is injective. One can verify using Proposition 3.3 and local coordinates that $\bar{\omega}$ is surjective. Thus the map $\bar{\omega}$ is a bundle isomorphism.

By taking exterior powers of the map $\bar{\omega}$, we can extend it to an isomorphism

$$\bar{\omega} : \wedge^p TM \rightarrow \wedge^p(\mathcal{R}^*)$$

and hence a $\mathcal{C}^\infty(M)$ -linear isomorphism

$$\bar{\omega} : \mathcal{V}^p(M) \rightarrow {}^{\mathcal{R}}\Omega^p(M).$$

Claim 5.8. *For any smooth multivector field η on a given smooth non-degenerate scattering Poisson manifold (M, Z, π) , we have $\bar{\omega}(d_\pi(\eta)) = -d(\bar{\omega}(\eta))$.*

Proof. We will proceed by induction on the degree of η and by using the Leibniz rule. Let η be a degree 0 form, that is $\eta \in \mathcal{C}^\infty(M)$. Then $\bar{\omega}(\eta) = \eta$ and $-d(\bar{\omega}(\eta)) = -d\eta$. Consider $d_\pi(\eta) = [\pi, \eta] = -X_\eta$, the Hamiltonian vector field of η . Thus $\bar{\omega}(-X_\eta) = -i_{X_\eta}\omega = -d\eta$. If $\eta = d_\pi f$ is an exact 1-vector field, then $\bar{\omega}(d_\pi(d_\pi f)) = \bar{\omega}(0) = 0$ and $d(\bar{\omega}(d_\pi f)) = d(\bar{\omega}(X_f)) = d(df) = 0$. By the Leibniz rule, the statement is true for all multivector fields. \square

Thus the claim shows that, up to a sign, the map $\bar{\omega}$ is an isomorphism that intertwines the differential operator d of the rigged algebroid \mathcal{R} de Rham complex with the differential operator d_π of the Lichnerowicz complex. Hence $\bar{\omega} : H_\pi^p(M) \rightarrow {}^{\mathcal{R}}H^p(M)$ is an isomorphism. \square

Given a $2n$ -dimensional non-degenerate scattering Poisson manifold (M, Z, π) , let ξ denote the contact distribution on Z induced by π and let $\Omega_\xi^k(Z)$ denote degree- k forms σ on Z such that for all $p \in Z$, σ_p is supported in $\wedge^k \xi_p$. That is,

$$\Omega_\xi^k(Z) := \left\{ \sigma \in \Omega^k(Z) \mid \forall p \in Z, \text{supp}(\sigma_p) \subseteq \wedge^k \xi_p \right\}.$$

Let α be a contact form on Z such that $\ker \alpha = \xi$. Note that

$$\mathcal{K}^k := \ker(d\alpha \wedge : \Omega_\xi^k(Z) \rightarrow \Omega_\xi^{k+2}(Z))$$

is independent of the choice of α because any other choice of contact form will give a symplectic structure conformal to $d\alpha$ on ξ . We adopt the convention that $\mathcal{K}^k = 0$ for $k \leq 0$.

In the following theorem, we will show that the cohomology class of $\mu \in {}^{\mathcal{R}}H^k(M)$ is uniquely determined by a smooth b -form and a 1-jet at Z of a closed form in ${}^{\mathcal{R}}\Omega^k(M)$. Let $\mathcal{J}_Z^1({}^{\mathcal{R}}\Omega_{cl}^k(M))$ denote the 1-jets at Z of closed forms in ${}^{\mathcal{R}}\Omega^k(M)$.

Theorem 5.9. *Given (M, Z, π) a $2n$ -dimensional non-degenerate scattering Poisson manifold, let ξ denote the contact distribution on Z induced by π . The Poisson cohomology $H_\pi^p(M)$ of (M, Z, π) is*

$$H^p(M) \oplus H^{p-1}(Z) \oplus \mathcal{J}_Z^1({}^{\mathcal{R}}\Omega_{cl}^p(M)).$$

Given a fixed Z defining function x ,

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus \Omega^{p-1}(Z) \oplus \Omega_\xi^{p-1}(Z) \oplus \mathcal{K}^{p-2}.$$

Remark 5.10. Given a contact hypersurface (Z, α) of dimension $2n + 1$, the map $d\alpha \wedge : \Omega_\xi^k(Z) \rightarrow \Omega_\xi^{k+2}(Z)$, by a local computation at a point, is injective for $k = 0, \dots, n - 1$ and is identically equal to zero for $k = 2n, 2n + 1$. Thus $\mathcal{K}^k = 0$ for $k = 0, \dots, n - 1$ and $\mathcal{K}^k = \Omega_\xi^k(Z)$ for $k = 2n, 2n + 1$.

Proof. We are left to compute $\mathcal{R}H^p(M)$. We have a short exact sequence

$$0 \rightarrow {}^b\Omega^k(M) \xrightarrow{i^*} \mathcal{R}\Omega^k(M) \xrightarrow{P} \mathcal{C}^k \rightarrow 0$$

where

$$\mathcal{C}^p = \mathcal{R}\Omega^p(M) / {}^b\Omega^p(M)$$

is the quotient under the inclusion bundle map $i : \mathcal{R}TM \rightarrow {}^bTM$. We have a differential $\mathcal{C}d$ induced by the differential $\mathcal{R}d$ on $\mathcal{R}\Omega^p(M)$. In particular, if P is the projection $\mathcal{R}\Omega^p(M) \rightarrow \mathcal{R}\Omega^p(M) / {}^b\Omega^p(M)$, then $\mathcal{C}d(\eta) = P(\mathcal{R}d(\theta))$ where $\theta \in \mathcal{R}\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $(\mathcal{C}d)^2 = 0$ and $(\mathcal{C}^*, \mathcal{C}d)$ is in fact a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , we can write a degree- k form ν in $\mathcal{R}\Omega^k(M)$ as an expansion in x of the form

$$\nu = \mu_b + \frac{dx}{x^{2k+1}} \wedge \left(\sum_{i=0}^{2k-1} \eta_i x^i \right) + \frac{1}{x^{2k}} \sum_{i=0}^{2k-1} \beta_i x^i + \frac{dx \wedge \alpha \wedge \theta}{x^{2k+2}} + \frac{\alpha \wedge \gamma}{x^{2k+1}},$$

μ_b is a b-form, α is the contact form on Z induced by ω and x , $\eta_i \in \Omega^{k-1}(Z)$, $\beta_i \in \Omega^k(Z)$, $\theta \in \Omega^{k-2}(Z)$ such that

$$\text{supp}(\theta) \subseteq \wedge^{k-2}\xi,$$

and $\gamma \in \Omega^{k-1}(Z)$ such that

$$\text{supp}(\gamma) \subseteq \wedge^{k-1}\xi.$$

We write $R_b(\nu) = \mu_b$ and $S_b(\nu) = \nu - R_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $R_b(\mathcal{R}d\nu) = \mathcal{R}d(R_b(\nu))$ and $S_b(\mathcal{R}d\nu) = \mathcal{R}d(S_b(\nu))$. Thus the trivialization τ induces a splitting $\mathcal{R}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence $\mathcal{R}H^k(M) = {}^bH^k(M) \oplus H^k(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

We have that

$$\mathcal{R}d(S_b(\nu)) = - \sum_{i=0}^{2k-1} \frac{dx}{x^{2k+1}} \wedge d\eta_i x^i - \sum_{i=0}^{2k-1} \frac{(2k-i)dx}{x^{2k+1}} \wedge \beta_i x^i + \sum_{i=0}^{2k-1} \frac{d\beta_i}{x^{2k}} x^i$$

$$-\frac{dx \wedge d\alpha \wedge \theta}{x^{2k+2}} + \frac{dx \wedge \alpha \wedge d\theta}{x^{2k+2}} - \frac{2k+1}{x^{2k+2}} dx \wedge \alpha \wedge \gamma + \frac{d\alpha \wedge \gamma}{x^{2k+1}} - \frac{\alpha \wedge d\gamma}{x^{2k+1}}.$$

Thus the kernel $\mathcal{C}d : \mathcal{C}^k \rightarrow \mathcal{C}^{k+1}$ is defined by the relation

$$-d\eta_i x^{i+1} - (2k-i)\beta_i x^{i+1} - d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \gamma = 0.$$

In order for the expression to be zero, the coefficients of the polynomial must be zero and thus

$$\beta_i = \frac{-d\eta_i}{(2k-i)}$$

for $i = 0, \dots, 2k-1$.

Now we consider the kernel relation given by the coefficient

$$-d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \gamma = 0.$$

By contracting with R , the Reeb vector field associated to α , we recover

$$d\theta - \alpha \wedge i_R d\theta - (2k+1)\gamma = 0.$$

Thus

$$\gamma = \frac{d\theta - \alpha \wedge i_R d\theta}{(2k+1)}.$$

Substituting this into the original expression, we have that

$$-d\alpha \wedge \theta + \alpha \wedge d\theta - (2k+1)\alpha \wedge \frac{d\theta}{(2k+1)} = 0$$

since $\alpha^2 = 0$. Thus $d\alpha \wedge \theta = 0$.

Thus all closed forms in \mathcal{C}^k are of the form

$$\frac{dx}{x^{2k+1}} \wedge \left(\sum_{i=0}^{2k-1} \eta_i x^i \right) + \frac{1}{x^{2k}} \sum_{i=0}^{2k-1} \frac{-d\eta_i}{(2k-i)} x^i + \frac{dx \wedge \alpha \wedge \theta}{x^{2k+2}} + \frac{\alpha}{x^{2k+1}} \wedge \frac{(d\theta - \alpha \wedge i_R d\theta)}{(2k+1)},$$

where $\theta \in \ker(d\alpha \wedge : \Omega_\xi^{k-2}(Z) \rightarrow \Omega_\xi^k(Z))$.

Elements in $\mathcal{C}d(\mathcal{C}^{k-1})$ are of the form

$$-\sum_{i=0}^{2k-3} \frac{dx}{x^{2k-1}} \wedge d\eta_i x^i - \sum_{i=0}^{2k-3} \frac{(2k-2-i)dx}{x^{2k-1}} \wedge \beta_i x^i + \sum_{i=0}^{2k-3} \frac{d\beta_i}{x^{2k-2}} x^i \quad (5.2)$$

$$-\frac{dx \wedge d\alpha \wedge \theta}{x^{2k}} + \frac{dx \wedge \alpha \wedge d\theta}{x^{2k}} - \frac{2k-1}{x^{2k}} dx \wedge \alpha \wedge \gamma + \frac{d\alpha \wedge \gamma}{x^{2k-1}} - \frac{\alpha \wedge d\gamma}{x^{2k-1}}.$$

Thus there is the element

$$\sum_{j=2}^{2k-1} \frac{-x^{j-2}\eta_j}{(2k-j)x^{2k-2}}$$

in \mathcal{C}^{k-1} such that

$$d \left(\sum_{j=2}^{2k-1} \frac{-x^{j-2} \eta_j}{(2k-j)x^{2k-2}} \right) = \sum_{j=2}^{2k-1} \left(\frac{dx}{x^{2k+1}} \wedge \eta_j x^j - \frac{d\eta_j x^j}{(2k-j)x^{2k}} \right).$$

If we express $\eta_i = \delta_i + \alpha \wedge \gamma_i$ for $\delta_i, \gamma_i \in \Omega_\xi^k(Z)$, then there is the element

$$\frac{-\alpha \wedge \gamma_1}{(2k-1)x^{2k-1}}$$

in \mathcal{C}^{k-1} such that

$$d \left(\frac{-\alpha \wedge \gamma_1}{(2k-1)x^{2k-1}} \right) = \frac{dx \wedge \alpha \wedge \gamma_1 x}{x^{2k+1}} - \frac{d(\alpha \wedge \gamma_1)x}{(2k-1)x^{2k}}.$$

By (5.2), the remaining terms in a closed form in \mathcal{C}^k are too singular to appear in the image $d(\mathcal{C}^{k-1})$. Thus an element of $H^k(\mathcal{C})$ has a representative of the form

$$\begin{aligned} \nu = & \frac{dx}{x^{2k+1}} \wedge (\delta_0 + \alpha \wedge \gamma_0) + \frac{dx}{x^{2k+1}} \wedge x\delta_1 + \frac{dx}{x^{2k+2}} \wedge \alpha \wedge \theta \\ & - \frac{d(\delta_0 + \alpha \wedge \gamma_0)}{(2k)x^{2k}} - \frac{d\delta_1}{(2k-1)x^{2k-1}} - \frac{d(\alpha \wedge \theta)}{(2k+1)x^{2k+1}} \end{aligned}$$

where $\delta_0, \gamma_0, \delta_1 \in \Omega_\xi^*(Z)$ and $\theta \in \mathcal{K}^{k-2} = \ker(d\alpha \wedge : \Omega_\xi^{k-2}(Z) \rightarrow \Omega_\xi^k(Z))$ and each such form represents a separate cohomology class. Thus for a fixed Z defining function x , by the map

$$\nu \mapsto (\delta_0 + \alpha \wedge \gamma_0, \delta_1, \theta),$$

$$H^k(\mathcal{C}) \simeq \Omega^{k-1}(Z) \oplus \Omega_\xi^{k-1}(Z) \oplus \mathcal{K}^{k-2}.$$

To conclude, we consider what happens under change of Z defining function x . Note that ν is completely determined by $i_{\partial x} \nu$. Further, note that representatives ν are equivalent to closed forms because the image $d(\mathcal{C}^{k-1})$ has empty intersection with the collection of forms of type ν .

Next, we can rearrange the dx coefficient in ν as

$$\underbrace{\frac{dx}{x^{2k+1}} \wedge \delta_0 + \frac{dx}{x^{2k+2}} \wedge \alpha \wedge \theta}_{0\text{-jet}} + \underbrace{\frac{dx}{x^{2k+1}} \wedge \alpha \wedge \gamma_0 x + \frac{dx}{x^{2k+1}} \wedge \delta_1 x}_{1\text{-jet}}.$$

Thus, $H^k(\mathcal{C})$ is in bijective correspondence with all 1-jets at Z of closed forms in ${}^{\mathcal{R}}\Omega^k(M)$, denoted $\mathcal{J}_Z^1({}^{\mathcal{R}}\Omega_{cl}^k(M))$.

We have shown that the Poisson cohomology $H_\pi^k(M)$ of (M, Z, π) is

$${}^b H^k(M) \oplus \mathcal{J}_Z^1({}^{\mathcal{R}}\Omega_{cl}^k(M))$$

and, given a fixed Z defining function x , is

$${}^bH^k(M) \oplus \Omega^{k-1}(Z) \oplus \Omega_\xi^{k-1}(Z) \oplus \mathcal{K}^{k-2}.$$

The final isomorphism is a consequence of the Mazzeo-Melrose theorem that ${}^bH^k(M) \simeq H^k(M) \oplus H^{k-1}(Z)$. \square

The rigged algebroid approach can also be used to compute the Poisson cohomology of other minimally degenerate Poisson structures, such as b and b^k -Poisson.

5.2.1. Poisson cohomology of a b^k -Poisson manifold. Let (M, Z, π) be a non-degenerate b^k -Poisson manifold and let ω be the corresponding b^k -symplectic form. Guillemin, Miranda, and Pires ([12], $k = 1$) and Scott ([25], $k \geq 2$) showed that ω induces a cosymplectic structure

$$(\theta, \eta) \in \Omega^1(Z) \times \Omega^2(Z)$$

on Z . That is, there exists a pair of closed forms such that

$$\theta \wedge \eta^{n-1} \neq 0$$

where the dimension of Z is $2n - 1$.

Let (M, Z, ω) be a b^k -symplectic manifold for $k \geq 2$. Following Scott, we will fix and work in a tubular neighborhood $Z \times (-\varepsilon, \varepsilon)_x$ where the form ω is expressible near Z as

$$\omega = \frac{dx}{x^k} \wedge \theta + \eta$$

where $Z = \{x = 0\}$ and (θ, η) is a cosymplectic structure on Z . However it suffices to only fix a finite jet of Z defining function, and for $k = 1$ it is unnecessary to fix anything.

Let R be the Reeb vector field associated to (θ, η) . That is the non-vanishing vector field R on Z such that

$$\theta(R) = 1 \text{ and } i_R \eta = 0.$$

Then TZ splits as

$$\mathbb{R}\langle R \rangle \oplus \mathbb{R}\langle \ker \theta \rangle.$$

We identify the b rigged algebroid \mathcal{R} as the algebroid whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; TM) : u|_Z \in \mathcal{C}^\infty(M; \ker \theta)\}.$$

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$ in Z such that

$$\theta = dr \text{ and } \eta = \sum_{i=1}^m ds_i \wedge dt_i,$$

the local sections of \mathcal{R} are smooth linear combinations of

$$x\partial x, x\partial r, \partial s_1, \partial t_1, \dots, \partial s_m, \partial t_m.$$

Notice that we can locally identify the dual elements of \mathcal{R}^* as

$$\frac{dx}{x}, \frac{\theta}{x}, ds_1, dt_1, \dots, ds_m, dt_m.$$

The b^k *rigged algebroid* \mathcal{R} is defined iteratively using the b -rigged algebroid ${}^b\mathcal{R}$. The b^2 -rigged algebroid is the vector bundle whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; {}^b\mathcal{R}) : u|_Z \in \mathcal{C}^\infty(M; \ker \theta)\}.$$

Given the b^{k-1} -rigged algebroid, the b^k -rigged algebroid is the the vector bundle whose space of sections is

$$\{u \in \mathcal{C}^\infty(M; {}^{b^{k-1}}\mathcal{R}) : u|_Z \in \mathcal{C}^\infty(M; \ker \theta)\}.$$

For local coordinates $(r, s_1, t_1, \dots, s_m, t_m)$ in Z such that

$$\theta = dr \text{ and } \beta = \sum_{i=1}^m ds_i \wedge dt_i,$$

the local sections of \mathcal{R} are smooth linear combinations of

$$x^k \partial x, x^k \partial r, \partial s_1, \partial t_1, \dots, \partial s_m, \partial t_m.$$

Notice that we can locally identify the dual elements of \mathcal{R}^* as

$$\frac{dx}{x^k}, \frac{\theta}{x^k}, ds_1, dt_1, \dots, ds_m, dt_m.$$

Lemma 5.11. *The Poisson cohomology of a non-degenerate b^k -Poisson manifold (M, Z, π) is isomorphic to the de Rham cohomology ${}^{\mathcal{R}}H^*(M)$ of the b^k rigged algebroid \mathcal{R} .*

The details of the proof of this Lemma are identical to those found in the proof of Lemma 5.7.

For $k \geq 2$, the de Rham cohomology of the b^k rigged algebroid will contain information about the symplectic foliation of the Poisson structure. The flow of the Reeb vector field associated to the cosymplectic structure defines a foliation on Z , which we will denote \mathcal{F}_R . Consider the horizontal forms on this foliation:

$$\Omega_h^p(Z) = \{\sigma \in \Omega^p(Z) \mid i_R \sigma = 0\}.$$

We define an exterior derivative

$$d_h = d - \theta \wedge \mathcal{L}_R.$$

First note that this is well defined on the complex: Indeed showing $d_h \sigma \in \Omega_h^*(Z)$ is equivalent to showing that $i_R d_h \sigma = 0$. Further $\sigma \in \Omega_h^*(Z)$ means that $i_R \sigma = 0$ and thus $\mathcal{L}_R \sigma = i_R d \sigma$. Thus

$$i_R(d_h \sigma) = i_R(d \sigma - \theta \wedge i_R d \sigma) = i_R d \sigma - i_R d \sigma = 0.$$

Next we will show that d_h squares to zero. Given $\sigma \in \Omega_h^p(Z)$,

$$\begin{aligned} d_h^2 \sigma &= d^2 \sigma - d(\theta \wedge \mathcal{L}_R \sigma) - \theta \wedge \mathcal{L}_R d\sigma + \theta \wedge \mathcal{L}_R(\theta \wedge \mathcal{L}_R \sigma) \\ &= \theta \wedge d\mathcal{L}_R \sigma - \theta \wedge \mathcal{L}_R d\sigma + \theta \wedge \mathcal{L}_R \theta \wedge \mathcal{L}_R \sigma = 0. \end{aligned}$$

Thus $(\Omega_h^*(Z), d_h)$ is a complex. We call its cohomology groups $H_h^*(\mathcal{F}_R)$ the *horizontal foliation cohomology of \mathcal{F}_R* .

While computing the Poisson cohomology of a non-degenerate b^k -Poisson manifold for all k , we recover the result of result of Ioan Mărcuț and Boris Osorno Torres [16] when $k = 1$.

Theorem 5.12. *Given a $2n$ -dimensional non-degenerate b^k -Poisson manifold (M, Z, π) , the Poisson cohomology $H_\pi^p(M)$ is*

$$H^p(M) \oplus H^{p-1}(Z) \text{ for } k = 1$$

$$H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1} \text{ for } k \geq 2.$$

Proof. All that remains is to compute ${}^{\mathcal{R}}H^p(M)$. The inclusion bundle map $i : {}^{\mathcal{R}}TM \rightarrow {}^bTM$ induces an inclusion of complexes

$$0 \rightarrow {}^b\Omega^p(M) \xrightarrow{i^*} {}^{\mathcal{R}}\Omega^p(M) \xrightarrow{P} \mathcal{C}^p \rightarrow 0$$

where

$$\mathcal{C}^p = {}^{\mathcal{R}}\Omega^p(M) / {}^b\Omega^p(M)$$

is the quotient. We have a differential ${}^{\mathcal{C}}d$ induced by the differential ${}^{\mathcal{R}}d$ on ${}^{\mathcal{R}}\Omega^p(M)$. In particular, if P is the projection ${}^{\mathcal{R}}\Omega^p(M) \rightarrow {}^{\mathcal{R}}\Omega^p(M) / {}^b\Omega^p(M)$, then ${}^{\mathcal{C}}d(\eta) = P({}^{\mathcal{R}}d(\theta))$ where $\theta \in {}^{\mathcal{R}}\Omega^p(M)$ is any form such that $P(\theta) = \eta$. Hence $({}^{\mathcal{C}}d)^2 = 0$ and $(\mathcal{C}^*, {}^{\mathcal{C}}d)$ is a complex.

Given a tubular neighborhood $\tau = Z \times (-\varepsilon, \varepsilon)_x$ of M near Z , we can write a degree k form μ in ${}^{\mathcal{R}}\Omega^k(M)$ as

$$\mu = \nu_b + \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge L_i x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (\theta \wedge M_i + N_i) x^i + \sum_{i=0}^{k-1} \frac{\theta}{x^k} \wedge P_i x^i$$

for ν_b a smooth b-form, and $L_i, M_i, N_i, P_i \in \Omega_h^*(Z)$.

We write $R_b(\nu) = \mu_b$ and $S_b(\nu) = \nu - R_b(\nu)$ for ‘regular’ and ‘singular’ parts. It is easy to see that $R_b({}^{\mathcal{R}}d\nu) = {}^{\mathcal{R}}d(R_b(\nu))$ and $S_b({}^{\mathcal{R}}d\nu) = {}^{\mathcal{R}}d(S_b(\nu))$. Thus the trivialization τ induces a splitting ${}^{\mathcal{R}}\Omega^*(M) = {}^b\Omega^*(M) \oplus \mathcal{C}^*$ as complexes. As a consequence ${}^{\mathcal{R}}H^p(M) = {}^bH^p(M) \oplus H^p(\mathcal{C}^*)$ and we are left to compute the cohomology of the quotient complex.

After identifying $\mathcal{C}^p = \{\mu \in \mathcal{R}\Omega^k(M) : \nu_b = 0\}$, the differential is given by

$$\begin{aligned} d\mu = & \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge dL_i x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (\theta \wedge dM_i - dN_i) x^i \\ & - \sum_{i=0}^{k-1} \frac{(k-i)dx}{x^{k+1}} \wedge \theta \wedge P_i x^i - \sum_{i=0}^{k-1} \frac{\theta}{x^k} \wedge dP_i x^i. \end{aligned}$$

Then $d\nu = 0$ if and only if

- $\theta \wedge dL_i = 0$ for $i = 0, \dots, k-2$
- $\theta \wedge dL_{k-1} - \theta \wedge kP_0 = 0$
- $-(k-i)\theta \wedge P_i + \theta \wedge dM_{i-1} - dN_{i-1} = 0$ for $i = 1, \dots, k-1$.

Note that there is $\tilde{\mu} \in \mathcal{C}^{p-1}$ of the form

$$\tilde{\mu} = \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge l_i x^i + \sum_{i=0}^{k-2} \frac{\theta}{x^k} \wedge \frac{M_i x^{i+1}}{(-k+i+1)} + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge n_i x^i - \frac{\theta}{kx^k} \wedge L_{k-1}$$

satisfying

$$\begin{aligned} d\tilde{\mu} = & \sum_{i=0}^{k-1} \frac{dx}{x^{2k}} \wedge \theta \wedge dl_i x^i + \sum_{i=0}^{k-2} \left(\frac{dx}{x^k} \wedge \theta \wedge M_i x^i - \frac{\theta}{x^k} \wedge \frac{dM_i x^{i+1}}{(-k+i+1)} \right) \\ & + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge dn_i x^i - \frac{dx}{x^{k+1}} \wedge \theta \wedge L_{k-1} + \frac{\theta}{kx^k} \wedge dL_{k-1}. \end{aligned}$$

Thus $[\mu - d\tilde{\mu}] \in H^p(\mathcal{C})$ has a representative

$$\sum_{i=0}^{k-2} \frac{dx}{x^{2k}} \wedge \theta \wedge (L_i - dl_i) x^i + \sum_{i=0}^{k-2} \frac{dx}{x^k} \wedge (N_i - dn_i) x^i + \sum_{i=1}^{k-1} \frac{1}{x^k} \frac{dN_{i-1}}{(k-i)} x^i$$

where $L_i, l_i, N_i, n_i \in \Omega_h^*(Z)$. Note that $d_h N_i \equiv 0$ since $dN_{i-1} = (-k+i)\theta \wedge P_i$.

Notice if two forms ν_1, ν_2 are representatives of the same cohomology class in $H^p(\mathcal{C})$, then the coefficients of the expression $\nu_1 - \nu_2$ must be exact. Thus, we have shown

$$H^p(\mathcal{C}) = \frac{\{L_i \in \Omega_h^{p-2}(Z) : d_h L_i = 0\}}{\{L_i : L_i = d_h l_i, l_i \in \Omega_h^{p-3}(Z)\}} \oplus \frac{\{N_i \in \Omega_h^{p-1}(Z) : d_h N_i = 0\}}{\{N_i : N_i = d_h n_i, n_i \in \Omega_h^{p-2}(Z)\}}$$

and

$$H_\pi^p(M) \simeq H^p(M) \oplus H^{p-1}(Z) \oplus (H_h^{p-2}(\mathcal{F}_R))^{k-1} \oplus (H_h^{p-1}(\mathcal{F}_R))^{k-1}.$$

□

Remark 5.13. It may seem surprising to some readers that the Poisson cohomology involves the horizontal forms of the foliation, rather than the basic forms.

Given the foliation \mathcal{F}_R , the basic forms are

$$\Omega_{b^2}^p(Z) = \{\sigma \in \Omega_h^p(Z) : \mathcal{L}_R \sigma = 0\}.$$

We will provide an example of a b^2 Poisson manifold to help convince readers of this fact.

Example 5.14. Let $M = \mathbb{R} \times \mathbb{T}^3$. Let x be the coordinate on \mathbb{R} and let θ_1, θ_2 , and θ_3 be the respective angular coordinates on the three copies of \mathbb{S}^1 in \mathbb{T}^3 . We define a b^2 -symplectic form on $\mathbb{R} \times \mathbb{T}^3$ as

$$\omega = \frac{dx}{x^2} \wedge d\theta_1 + d\theta_2 \wedge d\theta_3.$$

The singular hypersurface is $Z = \{0\} \times \mathbb{T}^3$. The symplectic form induces the cosymplectic structure $(d\theta_1, d\theta_2 \wedge d\theta_3)$ with Reeb vector field $\frac{\partial}{\partial \theta_1}$. The associated rigged algebroid \mathcal{R} is generated by

$$x^2 \frac{\partial}{\partial x}, x^2 \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}, \text{ and } \frac{\partial}{\partial \theta_3}.$$

Consider the rigged form

$$\nu = \frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge \cos(\theta_1) d\theta_3.$$

The one form $\cos(\theta_1) d\theta_3$ is a horizontal form because

$$i_{\frac{\partial}{\partial \theta_1}} \cos(\theta_1) d\theta_3 = 0.$$

Further, this form is closed under the horizontal differential because

$$\begin{aligned} d_h \cos(\theta_1) d\theta_3 &= (d - d\theta_1 \wedge \mathcal{L}_{\frac{\partial}{\partial \theta_1}}) \cos(\theta_1) d\theta_3 \\ &= -\sin(\theta_1) d\theta_1 \wedge d\theta_3 - d\theta_1 \wedge (i_{\frac{\partial}{\partial \theta_1}} (-\sin(\theta_1) d\theta_1 \wedge d\theta_3)) = 0. \end{aligned}$$

However, $\cos(\theta_1) d\theta_3$ is not a basic form because

$$\mathcal{L}_{\frac{\partial}{\partial \theta_1}} \cos(\theta_1) d\theta_3 = i_{\frac{\partial}{\partial \theta_1}} (-\sin(\theta_1) d\theta_1 \wedge d\theta_3) = -\sin(\theta_1) d\theta_3 \neq 0.$$

Consider

$$d_{\mathcal{R}} \nu = d \left(\frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge \cos(\theta_1) d\theta_3 \right) = \frac{dx}{x^2} \wedge \frac{d\theta_1}{x^2} \wedge [-\sin(\theta_1) d\theta_1 \wedge d\theta_3] = 0$$

and thus $[\nu] \in H_h^3(\mathbb{T}^3)$. Further, we can identify the class $[\nu]$ with $[\cos(\theta_1) d\theta_3] \in H_h^3(\mathbb{T}^3)$.

We will conclude with an example of a b^k Poisson manifold and give an explicit expression of its cohomology.

Example 5.15. Poisson cohomology of the b^k manifold $(\mathbb{T}^{2n}, \cup_2 \mathbb{T}^{2n-1})$

We will identify \mathbb{T}^{2n} with $\mathbb{S}^1 \times \mathbb{S}^1 \times \mathbb{T}^{2n-2}$. Let $\theta \in [0, 2\pi)$ be the angular coordinate on the first copy of \mathbb{S}^1 , let $d\phi$ be the volume form on the second \mathbb{S}^1 , and let β be the standard symplectic form on \mathbb{T}^{2n-2} .

We define a b^k -symplectic form by

$$\omega = \frac{d\theta}{\sin^k \theta} \wedge d\phi + \beta.$$

The singular hypersurface of this structure is the disjoint union of $\{0\} \times \mathbb{T}^{2n-1}$ and $\{\pi\} \times \mathbb{T}^{2n-1}$. The symplectic foliation of the dual Poisson structure contains two open symplectic leaves: $\mathbb{T}^{2n} \setminus (\{0\} \times \mathbb{T}^{2n-1} \cup \{\pi\} \times \mathbb{T}^{2n-1})$. The hypersurfaces $\{0\} \times \mathbb{T}^{2n-1}$ and $\{\pi\} \times \mathbb{T}^{2n-1}$ are foliated by leaves of the form $\{cst.\} \times \mathbb{T}^{2n-2}$.

The induced cosymplectic structure is $(d\phi, \beta)$ and the associated Reeb vectorfield is $\frac{\partial}{\partial \phi}$. Consider

$$\Omega_h^p(\mathbb{T}^{2n-1}) = \{\sigma \in \Omega^p(\mathbb{T}^{2n-1}) \mid i_{\partial \phi} \sigma = 0\} = \mathcal{C}^\infty(\mathbb{S}^1; \Omega^p(\mathbb{T}^{2n-2}))$$

and $d_h = d_{\mathbb{T}^{2n-2}}$. Then $H_h^p(\mathcal{F}_{\partial \phi}) = \mathcal{C}^\infty(\mathbb{S}^1; H^p(\mathbb{T}^{2n-2}))$. By Theorem 5.12, $H_\pi^p(\mathbb{T}^{2n})$ is computable as

$$H^p(\mathbb{T}^{2n}) \oplus [H^{p-1}(\mathbb{T}^{2n-1}) \oplus (\mathcal{C}^\infty(\mathbb{S}^1; H^{p-2}(\mathbb{T}^{2n-2})) \oplus \mathcal{C}^\infty(\mathbb{S}^1; H^{p-1}(\mathbb{T}^{2n-2})))^{k-1}]^2$$

and we note that the factors coming from the singular hypersurface are squared because we have a disjoint union of two tori.

6. Appendix: Proofs of Technical Lemmas

Lemma 6.1. *If (M, ω) is a strong convex symplectic filling of (Z, ξ) , then for some $c > 0$, there exists a collar neighborhood $Z \times [0, c)_r$ of Z on which*

$$\omega = d(e^{-r} \alpha)$$

and, given the projection $p : Z \times [0, c) \rightarrow Z$, $\alpha = p^(\tilde{\alpha})$ for an $\tilde{\alpha}$ satisfying $\ker \tilde{\alpha} = \xi$.*

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Proof. By definition of convex strong symplectic filling, near Z there exists a nowhere vanishing vector field v transverse to Z such that $\mathcal{L}_{-v}\omega = \omega$. By exponentiating v with respect to any Riemannian metric, we can choose a collar neighborhood $Z \times [0, c)_r$ of Z such that r is the coordinate for $[0, c)$, $c > 0$, and

$$v|_{Z \times [0, c)} = \frac{\partial}{\partial r}.$$

It follows from the definition of strong symplectic filling that

$$(i_{-\frac{\partial}{\partial r}}\omega)|_Z = \alpha$$

for some contact form α defining ξ on Z . Given the projection

$$p : Z \times [0, c) \rightarrow Z,$$

let $\tilde{\alpha} = p^*\alpha$. Let $\gamma = i_{-\frac{\partial}{\partial r}}\omega$. Then because $\mathcal{L}_{-v}\omega = \omega$, we have $d\gamma = \omega$.

Let x_1, x_2, \dots, x_n be any set of coordinates in Z . Then in these coordinates,

$$\gamma = gdr + \sum_i f_i dx_i$$

for some smooth functions $g, f_i \in \mathcal{C}^\infty(Z \times [0, c))$. We can compute

$$d\gamma = \sum_j \left(\frac{\partial g}{\partial x_j} dx_j \wedge dr + \sum_i \frac{\partial f_i}{\partial x_j} dx_j \wedge dx_i \right) + \sum_i \frac{\partial f_i}{\partial r} dr \wedge dx_i.$$

Then $i_{-\frac{\partial}{\partial r}}d\gamma = \gamma$ gives the relations

$$\sum_i \left(\frac{\partial g}{\partial x_i} - \frac{\partial f_i}{\partial r} \right) dx_i = gdr + \sum_i f_i dx_i.$$

In other words,

$$g = 0, \text{ and } -\frac{\partial f_i}{\partial r} = f_i.$$

Thus $f_i = c_i \cdot e^{-r}$ for some constants c_i . Since $\gamma|_{\{r=0\}} = \alpha$, we have that

$$\gamma = e^{-r}\tilde{\alpha} \text{ and } \omega = d(e^{-r}\tilde{\alpha}).$$

A similar computation shows the statement for a concave filling. \square

Lemma 6.2. *There exists a smooth function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, ϕ supported in $(1/2, 2)$, and a smooth function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, ψ supported in $(-\infty, 1)$ with $\psi|_{(-\infty, 7/8)} \equiv 1$ such that ω as in equation (4.1) is non-degenerate.*

Proof. Let

$$\phi(r) = \begin{cases} e^{\frac{r}{(r-1/2)(r-2)}} & \text{if } r \in (1/2, 2); \\ 0 & \text{otherwise.} \end{cases} \quad (6.1)$$

and

$$\psi(r) = \begin{cases} 1 & \text{if } r \leq 7/8; \\ 1 - \frac{e^{\frac{-1}{(r-1)}}}{e^{\frac{-1}{(r-1)}} + e^{\frac{-1}{(7/8-r)}}} & \text{if } r \in (7/8, 1); \\ 0 & \text{if } r \geq 1. \end{cases} \quad (6.2)$$

Let

$$\omega = d \left(\left(\frac{\phi(r_1)}{(r_1-1)^2} + \psi(r_1) \right) \tilde{\gamma} \right) + d \left(\left(\frac{\phi(r_2)}{(r_2-1)^2} + \psi(r_2) \right) \gamma \right)$$

where $r_1 r_2 = 1$, $\tilde{\gamma} = \frac{e^{-r_2}}{e^{-r_1}} \gamma$, and $\gamma = e^{-r_1} \alpha$.

By symmetry of expression, the following are equivalent:

- verifying non-degeneracy on $Z \times (1/2, 2)_{r_1}$ when $r_1 - 1 < 0$
- verifying non-degeneracy on $Z \times (1/2, 2)_{r_2}$ when $r_2 - 1 < 0$.

Further, because $\phi(r)$ is multiplicatively symmetric on the interval $(1/2, 2)$ about 1, the following are equivalent:

- verifying that ω is non-degenerate on $Z \times (1/2, 2)_{r_1}$ when $r_1 - 1 < 0$
- verifying that ω is non-degenerate on $Z \times (1/2, 2)_{r_2}$ when $r_2 - 1 > 0$.

Thus to show ω is non-degenerate on all of $M_1 \cup_Z M_2$, it suffices to check that ω is non-degenerate in coordinate r_1 when $r_1 - 1 < 0$.

$$\begin{aligned} \omega &= \left[\frac{\phi'(r_1)}{(r_1-1)^2} - 2 \frac{\phi(r_1)}{(r_1-1)^3} + \psi'(r_1) \right] dr_1 \wedge \tilde{\gamma} + \left[\frac{\phi(r_1)}{(r_1-1)^2} + \psi(r_1) \right] d\tilde{\gamma} \\ &+ \left[\frac{\phi'(r_2)}{(r_2-1)^2} - 2 \frac{\phi(r_2)}{(r_2-1)^3} + \psi'(r_2) \right] dr_2 \wedge \gamma + \left[\frac{\phi(r_2)}{(r_2-1)^2} + \psi(r_2) \right] d\gamma. \end{aligned}$$

Since $r_2 = \frac{1}{r_1}$, we can compute $dr_2 = \frac{-1}{r_1^2} dr_1$.

Note that

$$\tilde{\gamma} = \frac{e^{-r_2}}{e^{-r_1}} \gamma = e^{-1/r_1 + r_1} \gamma.$$

Thus

$$dr \wedge \tilde{\gamma} = e^{-1/r_1 + r_1} dr \wedge \gamma$$

and

$$d\tilde{\gamma} = e^{-1/r_1 + r_1} \left(\frac{1}{r_1^2} + 1 \right) dr \wedge \gamma + e^{-1/r_1 + r_1} d\gamma.$$

Then

$$\begin{aligned} & \left[\frac{\phi'(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^2} - 2\frac{\phi(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^3} + \psi' \left(\frac{1}{r_1} \right) \right] d \left(\frac{1}{r_1} \right) \wedge \gamma + \left[\frac{\phi(\frac{1}{r_1})}{(\frac{1}{r_1} - 1)^2} + \psi \left(\frac{1}{r_1} \right) \right] d\gamma \\ &= \left[-\frac{\phi' \left(\frac{1}{r_1} \right)}{(r_1 - 1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r_1 - 1)^3} - \frac{\psi' \left(\frac{1}{r_1} \right)}{r_1^2} \right] dr_1 \wedge \gamma + \left[\frac{\phi(\frac{1}{r_1})r_1^2}{(r_1 - 1)^2} + \psi \left(\frac{1}{r_1} \right) \right] d\gamma. \end{aligned}$$

Thus $\omega = A dr_1 \wedge \gamma + B d\gamma$, where

$$\begin{aligned} A &= e^{-1/r_1+r_1} \frac{\phi'(r_1)}{(r_1 - 1)^2} - 2e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1 - 1)^3} + e^{-1/r_1+r_1} \psi'(r_1) - \frac{\psi'(\frac{1}{r_1})}{r_1^2} + \\ & \left(\frac{1}{r_1^2} + 1 \right) e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1 - 1)^2} + \left(\frac{1}{r_1^2} + 1 \right) e^{-1/r_1+r_1} \psi(r_1) - \frac{\phi'(\frac{1}{r_1})}{(r_1 - 1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r_1 - 1)^3}, \end{aligned}$$

and

$$B = e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1 - 1)^2} + e^{-1/r_1+r_1} \psi(r_1) + \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1 - 1)^2} + \psi \left(\frac{1}{r_1} \right).$$

Then

$$\omega^n = AB^{n-1} dr_1 \wedge \gamma \wedge (d\gamma)^{n-1} + B^n (d\gamma)^n.$$

Notice

$$(d\gamma)^n = -e^{-r_1 n} dr_1 \wedge \alpha \wedge (d\alpha)^{n-1}$$

and

$$dr_1 \wedge \gamma \wedge (d\gamma)^{n-1} = e^{-r_1 n} dr_1 \wedge \alpha \wedge (d\alpha)^{n-1}.$$

Thus we are left to show that $AB^{n-1} - B^n > 0$. Notice that $B > 0$. So we will show that $A - B > 0$.

$$\begin{aligned} A - B &= e^{-1/r_1+r_1} \left(\frac{\phi'(r_1)}{(r_1 - 1)^2} - 2\frac{\phi(r_1)}{(r_1 - 1)^3} + \psi'(r_1) \right) + \frac{1}{r_1^2} e^{-1/r_1+r_1} \frac{\phi(r_1)}{(r_1 - 1)^2} \\ &+ \frac{1}{r_1^2} e^{-1/r_1+r_1} \psi(r_1) - \frac{\phi'(\frac{1}{r_1})}{(r_1 - 1)^2} - \frac{2\phi(\frac{1}{r_1})r_1}{(r_1 - 1)^3} - \frac{\psi'(\frac{1}{r_1})}{r_1^2} - \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1 - 1)^2} - \psi \left(\frac{1}{r_1} \right). \end{aligned}$$

Notice

$$e^{-1/r_1+r_1} \left(\frac{\phi'(r_1)}{(r_1 - 1)^2} - 2\frac{\phi(r_1)}{(r_1 - 1)^3} + \psi'(r_1) \right) > 0$$

because on the interval $(7/8, 1)$ we have $\psi' \geq -128$ and

$$\frac{\phi'(r_1)}{(r_1 - 1)^2} - 2\frac{\phi(r_1)}{(r_1 - 1)^3} > 139.$$

Next, observe that

$$-\frac{2\phi(\frac{1}{r_1})r_1}{(r_1 - 1)^3} - \frac{\phi(\frac{1}{r_1})r_1^2}{(r_1 - 1)^2} > 0$$

because

$$-\frac{2r_1}{(r_1 - 1)^3} - \frac{r_1^2}{(r_1 - 1)^2} > 0$$

for $r_1 \in (1/2, 1)$.

Finally, notice

$$-\frac{\phi'(\frac{1}{r_1})}{(r_1 - 1)^2}, \frac{1}{r_1^2}e^{-1/r_1+r_1}\frac{\phi(r_1)}{(r_1 - 1)^2}, \text{ and } \frac{1}{r_1^2}e^{-1/r_1+r_1}\psi(r_1)$$

are positive while

$$-\frac{\psi'(\frac{1}{r_1})}{r_1^2} \text{ and } \psi(\frac{1}{r_1})$$

are zero for $r_1 - 1 < 0$. □

Lemma 6.3. *There exists a smooth function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, ψ supported in $(-2, \infty)$ with $\psi|_{(-1, \infty)} \equiv 1$ such that ω as in equation (4.2) is non-degenerate as a folded symplectic form.*

Proof. We define $\psi(r)$ as

$$\psi(r) = \begin{cases} 0 & \text{if } r \leq -2; \\ \frac{e^{\frac{-1}{(r+2)}}}{e^{\frac{-1}{(r+2)}} + e^{\frac{-1}{(-1-r)}}} & \text{if } r \in (-2, -1); \\ 1 & \text{if } r \geq -1. \end{cases} \quad (6.3)$$

Let

$$\omega = d(\psi(r_1)e^{r_1}\alpha + \psi(r_2)e^{r_2}\alpha)$$

where $r_1 + r_2 = 0$. By the symmetry of this expression, to show ω is non-degenerate on all of $(M_1 \cup_Z M_2) \setminus Z$, it suffices to check that ω is non-degenerate when $r_1 > 0$. We have,

$$\begin{aligned} \omega &= (e^{r_1}\psi'(r_1) + e^{r_1}\psi(r_1) - e^{-r_1}\psi'(-r_1) - e^{-r_1}\psi(-r_1)) dr_1 \wedge \alpha \\ &\quad + (\psi(r_1)e^{r_1} + \psi(-r_1)e^{-r_1}) d\alpha. \end{aligned}$$

The non-degeneracy of ω will follow from checking that the coefficients of $d\alpha$ and $dr_1 \wedge \alpha$ are strictly positive. Note

$$\psi(r_1)e^{r_1} + \psi(-r_1)e^{-r_1}$$

is always positive. Since $\psi'(r_1) = 0$, we are left to check that

$$e^{r_1}\psi(r_1) - e^{-r_1}\psi'(-r_1) - e^{-r_1}\psi(-r_1) > 0.$$

Notice for $r_1 \in (1, 2)$, we have

$$\psi'(-r_1) < 3, \quad \psi(r_1) = 1, \quad \text{and} \quad \psi(-r_1) \leq 1.$$

Thus for $r_1 \in (1, 2)$, we must show

$$e^{r_1} - 4e^{-r_1} > 0.$$

In other words, $e^2 > 4$, which is true. For $r_1 \in (0, 1]$, note that

$$\psi'(-r_1) = 0, \quad \text{and} \quad \psi(r_1) = \psi(-r_1) = 1.$$

Thus the inequality is reduced to showing $e^{r_1} - e^{-r_1} > 0$. In other words for $r_1 \in (1, 2)$, we need $e^{2r_1} > 1$. But this inequality is true.

We will verify that ω is folded-symplectic using an equivalent definition of a folded symplectic form [3]: Let M be a $2d$ -dimensional manifold and $\omega \in \Omega^2(M)$ closed. Let Z be the set of points where $\omega^d = 0$. Let ω^d intersect the zero section of $\wedge^{2d}T^*M$ transversally. Let i be the inclusion map of Z into M . If the form $i^*\omega^{d-1} \in \Omega^{2d-2}(Z)$ is non-vanishing, ω is said to be a folded symplectic form.

Note $\omega|_Z = 2d\alpha$ and $\alpha \wedge d\alpha^{d-1}|_Z \neq 0$. Thus $d\alpha^{d-1}$ is non-vanishing and we have shown that ω is a folded symplectic structure. □

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